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S. J. Patterson

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DIOPHANTINE APPROXIMATION IN FUCHSIAN GROUPS

By S. J. PATTERSON
Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 Mill Lane, Cambridge

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CONTENTS

Let G be a finitely generated Fuchsian group of the first kind acting on the unit disk Δ and let S be the unit circle. If g is a Möbius transformation, represented by $\begin{pmatrix}\alpha & \beta \\ \bar{\beta} & \bar{\alpha}\end{pmatrix}$ $(|\alpha|^2-|\beta|^2=1)$ we write $\mu(g)=2(|\alpha|^2+|\beta|^2)$. Let $\zeta\in S$. Then there is a constant $c>0$ so that for any $\eta\in S$ which is not a parabolic fixed point of G the inequality

$$|\eta-g(\zeta)|<c/\mu(g)$$

has infinitely many solutions in G. This has been known for a long time (Hedlund's lemma).

This can be significantly sharpened in the following manner. If G has parabolic elements let ζ be a fixed parabolic fixed point, otherwise let it be a fixed hyperbolic fixed point of G. Then there is $c>0$ so that for any $X>2$, $\eta\in S$ there is a solution $g\in G$ of

$$\begin{aligned}|\eta-g(\zeta)| &\leqslant c/\surd(X\mu(g)) \quad (\zeta \text{ parabolic})\\ &\leqslant c/X \quad\quad\quad\quad (\zeta \text{ hyperbolic})\end{aligned}$$

with $\mu(g)\leqslant X$. From this we show that if $w(x)$ is a decreasing function satisfying $w(2x)/w(x)\geqslant c>0$ then the set

$$A=\{\eta\in S\colon |\eta-g(\zeta)|\leqslant w(\mu(g))/\mu(g) \text{ is soluble for infinitely many } g\in G\}$$

has measure 0 or 2π as $\Sigma w(2^n)$ converges or diverges. Finally we show that the set

$$B = \{\eta \in S\text{: there is } c > 0 \text{ so that } |\eta - g(\zeta)| > c/\mu(g) \text{ for all } g \in G\},$$

has Hausdorff dimension 1, although by the result above it has measure 0.

These results are analogous to various theorems in the metrical theory of numbers, and they reduce to these if G is taken to be the modular group. The proofs involve a close study of the geometry of the action of G on Δ.

1. Introduction

This paper is concerned with the geometric and measure-theoretic structure of the limit set of a Fuchsian group. By a Fuchsian group we shall understand a finitely generated Fuchsian group; we shall not attempt to investigate the pathologies of infinitely generated Fuchsian groups.

The present work splits into two parts. Up to § 7 we give a complete account of the geometry of the action of a Fuchsian group both on the open disk and on the unit circle. Although this has been studied in the past, the account given here is more detailed and systematic than anything in the literature. The detail, which at times may seem excessive, is required for applications in the second part.

The other part §§ 8–10, adopts the following point of view. The rational numbers can be characterized as the parabolic vertices of the modular group Γ. The theory of diophantine approximation (see, for example, Cassels 1965) gives ways of describing how well the rationals approximate a given number. The corresponding question for a Fuchsian group is: how well do the images of a distinguished point approximate an arbitrary limit point?

This problem has already been raised by (Rankin 1957) and (Lehner 1964), and to some extent answered by them. The first part of this paper contains a complete solution. In the second part we push the analogy further and seek theorems concerning the behaviour of almost all points – that is, corresponding to 'metric number theory'. In fact we can obtain results almost (but not quite) as sharp as their classical counterparts. This is carried out in § 9 and the structure of the exceptional set is described in § 10. Of course, this is only meaningful for groups of the first kind.

Fuchsian groups are usually denoted by G; only rarely shall we have to speak of more than one at a time. They shall usually act on the unit disk Δ; we shall only consider the action on the upper half-plane $\boldsymbol{H}$ when we want to examine some part of the group and can display it by an appropriate representation. For example, we will often examine a parabolic vertex by conjugating to $\boldsymbol{H}$ and making ∞ the parabolic vertex.

All Fuchsian groups will be finitely generated and non-elementary.

The limit set of G is denoted by L_G or $L(G)$. G_a is the subgroup of G fixing $a \in \overline{\Delta}$ and G_{ab} is the subgroup fixing $a, b \in \overline{\Delta}$.

con (D) will denote the group of conformal transformations of D. If $g \in \text{con}\,(\Delta)$ it is a bilinear map and can be represented by a matrix $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ $(|\alpha|^2 - |\beta|^2 = 1)$. We will write

$$\mu(g) = 2(|\alpha|^2 + |\beta|^2).$$

It is easy to see that
$$\mu(gh) \leqslant \mu(g)\,\mu(h).$$

The basic properties of μ may be found in (Beardon & Nicholls 1972).

The hyperbolic distance between a and b is denoted by $[a, b]$. Also we shall set

$$h(a,b) = \frac{|1-\bar{a}b|^2}{(1-|a|^2)(1-|b|^2)}.$$

Then one finds that $$1+\cosh[a,b] = 2h(a,b)$$
and $$\mu(g) = 4h(0,g(0))-2 = 2\cosh[0,g(0)].$$

S will represent the unit circle. $c_1, c_2, \ldots$ are numbers ('constants'), held fixed during an argument.

Theorems, etc., will be numbered inside a section. So theorem 8.2 will mean theorem 2 of § 8.

2. Geometry of the fundamental domain and the limit set

Let G be a Fuchsian group acting on Δ. Then there is an (open) fundamental domain D in Δ so that

(a) ∂D is a finite collection $\{u_j\}$ of geodesic arcs,

(b) $\{u_j\}$ splits uniquely into pairs u_k, u'_k in such a way that there is $\gamma_k \in G$ so that $u_k = \gamma_k(u'_k)$, the γ_k are distinct and generate G,

(c) $0 \in D$,

(d) if u_j, u_k meet at $p \in S$ then p is a parabolic vertex; $u_j = u'_k$ and γ_k generates G_p. No other u_l meets $G\{p\}$.

A justification of this can be found in Greenberg (1967).

L_G is a closed subset of S. Thus $\Omega = L_G^c (= S \setminus L_G)$ is open and so is a countable union of disjoint intervals, say $\Omega = \bigcup_j \Omega_j$. We shall say that Ω_j and Ω_k are *equivalent* if there is $g \in G$ so that $g\Omega_j = \Omega_k$. It is clear, as L_G is invariant under G, that either $g\Omega_j = \Omega_k$ or $g\Omega_j \cap \Omega_k = \varnothing$. Now we have

Theorem. *There are only a finite number of equivalence classes of Ω_j. There is a hyperbolic subgroup (but no larger subgroup), G_j say, preserving Ω_j.*

This is proved in Greenberg (1967) but it is not stated formally.

Let η_j, η'_j be the end-points of Ω_j and let λ_j be that arc of a circle lying in Δ, joining η_j, η'_j, making an internal angle $\alpha > 0$ with Ω_j. The collection of all λ_j is a figure invariant under G as G preserves $\{\Omega_j\}$, angles and orientation. Let $\Lambda_j = \Lambda_j(\alpha)$ be the *open* region between λ_j and Ω_j; it is lens-shaped and we call it an *α-lens*. As G is non-elementary the Ω_j have distinct end-points (consider the action of G_j on $S \setminus \Omega_j$) and the λ_j cannot intersect if $\alpha \leqslant \frac{1}{2}\pi$. In particular, if $\alpha \leqslant \frac{1}{2}\pi$ the $\Lambda_j(\alpha)$ are disjoint and are permuted by G. Let $K_G(\alpha) = \Delta \setminus \bigcup_j \Lambda_j(\alpha)$. If $\alpha \leqslant \frac{1}{2}\pi$, $K_G(\alpha)$ is hyperbolically convex.

The next point to note is that $D \cap K_G(\alpha)$ has finite (hyperbolic) area and if G has no parabolic elements $D \cap K_G(\alpha)$ is relatively compact. This follows as the only infinite parts of D are those adjacent to free sides (i.e. $\{\Omega_j \cap \bar{D}\}$) and cusps. Any free side is in some $\bar{\Lambda}_j$. All this is a direct consequence of the description of D given above.

Suppose now that G has parabolic elements and let $p_1, \ldots, p_r$ be the parabolic vertices lying on ∂D. We call an open disk contained in Δ and tangent to S at p a *horocycle* at p. Construct horocycles C'_j at p_j. Now refer this to $\boldsymbol{H}$ with $p_1 = \infty$. We know that the diameter of a horocycle $g(C'_j)$ ($g \in G$, $gp_j \neq \infty$) is bounded (Lehner 1964) and so we can find $C_1 \subseteq C'_1$, a horocycle at ∞ so that

(a) C_1 meets no image of $\{C'_j\}$ under G other than C'_1 (and hence no image of $\{C_1, C'_2, \ldots, C'_r\}$, other than C_1, under G),

(b) if $\alpha \leqslant \frac{1}{2}\pi$ C_1 meets no $\Lambda_k(\alpha)$.

(b) follows as the length of Ω_k is clearly bounded by the translation length of G_∞; hence the height of Λ_k is bounded.

We can repeat this argument to each j. Thus

Proposition 1. *There is a horocycle C_p at each parabolic vertex p and a $\frac{1}{2}\pi$-lens Λ_j on each Ω_j so that*

1. *$\{C_p, \Lambda_j\}$ are disjoint.*
2. *$C_{g(p)} = g(C_p)$.*
3. *$D \smallsetminus (\bigcup_p C_p \cup \bigcup_j \Lambda_j)$ is relatively compact in Δ.*
4. *D meets only a finite number of C_p and Λ_j.*

If p is any parabolic vertex then $p = g(p_j)$ for some j; then define $C_p = g(C_{p_j})$. Then the construction above is sufficient to imply the proposition.

This completes the construction but it is worth while adding a few words about the philosophy. We have cut D, or rather $G \smallsetminus \Delta$ into several distinct pieces which are associated with either a free side, or a parabolic vertex, or a compact subset of D. The first two have only an elementary group attached to them and are hardly more complicated than objects associated to those groups. The compact part is the most complicated in the sense that

$$\pi_1(G\backslash(K_G(\alpha) \smallsetminus \bigcup_p C_p)) = G$$

but it is greatly simplified in so far as it is compact. So we have separated physically distinct sources of difficulty. This method will be systematically employed. Incidentally, in this context parabolic vertices can be thought of as degenerate cases of intervals of discontinuity.

3. Hedlund's Lemma and a general approximation theorem

Fix a Fuchsian group G. If $0 < \beta \leqslant \frac{1}{2}\pi$ then we call an arc of a circle, in Δ, that meets S at an angle β a *β-line*; thus a geodesic is a $\frac{1}{2}\pi$-line.

Theorem 1 (*Hedlund's Lemma*). *Given α $(0 < \alpha \leqslant \frac{1}{2}\pi)$ there is a compact set K_α in Δ with the following property: if x is a non-parabolic limit point and λ is a β-line from $\zeta \in \Delta$ to x $(\alpha \leqslant \beta)$ then there is a sequence of points (x_n) on λ and (g_n) in G so that $x_n \to x$ and $g_n^{-1}(x_n) \in K_\alpha$.*

Futher K_α can be chosen independently of α if and only if G is of the first kind.

This is the classical theorem in the part of mathematics with which we are concerned. Although it is essentially due to Hedlund it was Lehner who made it really explicit; see Lehner (1964). Our proof is modelled on Lehner's.

Proof. Let us start with a purely geometric observation. If L is a line through 0 making an angle α with the positive half-line and C is a circle meeting $\boldsymbol{R}$ at an angle β where $\frac{1}{2}\pi \geqslant \beta > \alpha$ then either L and C do not meet in $\boldsymbol{H}$ *or* at least one of the intersections of C with $\boldsymbol{R}$ is on the positive half-line. The proof can be left to the reader.

If we transfer this to Δ and apply it to our situation we see that if the β-line λ meets some $\Lambda_j(\alpha)$ then λ can be extended to a β-line with an end-point in Ω_j. (To make the translation, map the positive half-line to Ω_j.) Consequently λ can meet at most one $\Lambda_j(\alpha)$, for otherwise both end-points would lie in Ω. So there is ζ' on λ so that the segment of λ between ζ' and x meets no $\Lambda_j(\alpha)$; call this segment λ'.

Now let $$K_\alpha = \overline{(D \cap K_G(\alpha)) \smallsetminus \bigcup_p C_p},$$

where the union is taken over all p although only a finite number of C_p meet D. The C_p here are as constructed in § 2. K_α is compact in Δ. Let $y \in \lambda'$. Then $y \in K_G(\alpha)$ and so there is $g \in G$ so that

$$g^{-1}(y) \in \overline{D \cap K_G(\alpha)}.$$

Suppose that $g^{-1}(y) \in C_p$. The part of λ' from y to x cannot lie in $g(C_p)$. So there is a point of λ' between y and x lying on $\partial(g(C_p))$. Thus there is $\gamma \in g(G_p)$ so that $\gamma^{-1}(\lambda')$ meets ∂K_α and so K_α, at the image of a point of λ' closer to x than y. Thus whether or not $y \in \bigcup C_p$ there is y' on λ' between y and x so that for $\gamma \in G$ $$\gamma^{-1}(y') \in \overline{D \cap K_G(\alpha) \smallsetminus \bigcup_p C_p} = K_\alpha.$$

This shows that there is a sequence (x_n) with the properties described in the theorem.

If G is of the first kind, $K_G(\alpha) = \Delta$ and K_α is independent of α. If G is of the second kind take x to be an end-point of Ω_j and λ to be the α-line $\partial \Lambda_j(\alpha) \cap \Delta$. In this case as α decreases the minimum (hyperbolic) distance of any image, under G, of λ from 0 tends to ∞; this is a consequence of the discussion of § 2. Thus no fixed compact set K can meet the images of λ for all α. This completes the proof.

Corollary (Beardon). *Let x be a non-parabolic limit point. There is a constant $c > 0$, independent of x, and a sequence (g_n) in G so that*

$$|g_n(0) - x| \leqslant c/\mu(g_n) \quad (\mu(g_n) \to \infty).$$

Proof. Let $\alpha, \beta = \frac{1}{2}\pi$ and let $\zeta = 0$. We construct the radius, i.e. $\frac{1}{2}\pi$-line, to x. By the theorem there is a constant r, independent of x, $x_n \to x$ along the radius, and $g_n \in G$ so that $|g_n^{-1}(x_n)| \leqslant r < 1$. Clearly $\mu(g_n) \to \infty$. Let $u_n = g_n^{-1}(x_n)$. Now $|x_n - x| = 1 - |x_n|$. But

$$\begin{aligned} |g_n(u_n) - g_n(0)| &= |g_n'(u_n)|^{\frac{1}{2}}\,|g_n'(0)|^{\frac{1}{2}}\,|u_n - 0| \\ &\leqslant c_1/\mu(g_n). \end{aligned}$$

Thus $$\begin{aligned} |g_n(0) - x| &\leqslant (1 - |x_n|) + (c_1/\mu(g_n)) \\ &\leqslant (1 - |g_n(0)|) + 2c_1/\mu(g_n). \end{aligned}$$

But $1 - |g_n(0)| \leqslant 4/\mu(g_n)$ and the corollary follows.

The following general approximation theorem contains almost all of the previously known results (with the exception of some of Rankin's theorems). If it is interpreted on $\boldsymbol{H}$ and applied to the modular group it gives Hurwitz's theorem (without an explicit constant).

Theorem 2. *Let x be a non-parabolic limit point and $y \in \overline{\Delta}$. Then there is $c > 0$ depending only on G so that $|x - g(y)| \leqslant c/\mu(g)$ can be solved for infinitely many $g \in G$.*

We need a lemma.

Lemma. *There is a finite open cover of L_G by intervals (I_j) $(1 \leqslant j \leqslant n)$ in S so that for each I_j there is $h_j \in G$ so that $d(I_j, h_j(I_j)) > 0$ (d is the Euclidean distance).*

Proof. As G is non-elementary for each $x \in L_G$ there is $h_x \in G$ so that $h_x(x) \neq x$. h is a diffeomorphism and so there is a neighbourhood I_x of x so that $d(I_x, h_x(I_x)) > 0$. $\{I_x\}$ covers L_G and as L_G is compact we can take a finite subcover, which does what is required.

Proof of theorem. By the lemma there is a finite subset of G, H_0 say, and a constant $c_1 > 0$ so that if $x \in L_G$ and $y \in \overline{\Delta}$ there is $h \in H_0$ so that $|h(x) - y| \geqslant c_1$. For we set $d = \min(d(I_j, h_j(I_j)))$ and then either $d(y, I_j) \geqslant \frac{1}{2}d$ or $d(y, h_j(I_j)) \geqslant \frac{1}{2}d$. So if we set $H_0 = \{I, h_1, \ldots, h_n\}$, $c_1 = \frac{1}{2}d$, they do what is required.

As $\mu(gh) \leqslant \mu(g)\,\mu(h)$, for $g \in G$, $h \in H_0$

$$c_2\mu(g) \leqslant \mu(gh^{-1}) \leqslant c_3\mu(g)$$

for suitable $c_2, c_3 > 0$. As in the corollary above there is c_4 so that, for $h \in H_0$,

$$|g_n(h^{-1}(0)) - g_n(0)| \leqslant c_4/\mu(g_n).$$

By that corollary, with the sequence (g_n) found there,

$$|g_n h^{-1}(0) - x| \leqslant c_5/\mu(g_n).$$

We may suppose that $\lim g_n^{-1}(\infty)$ exists; the sequence accumulates on S and so, at least on a subsequence, the limit exists. Let w be this limit. There is $h \in H_0$ so that $|h(w) - y| > c_1$ and so on a further subsequence $|hg_n^{-1}(\infty) - y| \geqslant \frac{1}{4}c_1$. If $u_n = g_n h^{-1}$,

$$|u_n(y) - u_n(0)| = \frac{4}{\mu(u_n)+2}\,\frac{1}{|y - u_n^{-1}(\infty)|} \leqslant \frac{4}{c_1\,\mu(u_n)}.$$

The theorem is now proved as

$$|u_n(y) - x| \leqslant c_5/\mu(g_n) + (4/c_1)/\mu(u_n) \leqslant (c_3c_5 + 4c_1^{-1})/\mu(u_n).$$

This theorem is very general; too much so to be of much use. Our efforts will now be directed towards a more useful result of the same type.

4. Further geometrical considerations

In order to carry out the programme indicated above it is necessary to study the parabolic and hyperbolic fixed points in some detail. The results and methods of this section form the basis for doing this and all that follows.

To begin, let $\eta, \eta' \in S$, $\eta \neq \eta'$. If $0 < \alpha < \frac{1}{2}\pi$ there are two distinct α-lines joining η, η'; let $C(\eta, \eta'\colon \alpha)$ be the open region in Δ trapped between these. If the reader finds this description unsatisfactory he is referred to the next section where he will find an analytic description. If we make the construction on $\boldsymbol{H}$, $C(0, \infty; \alpha)$ is the cone $\{\alpha < \arg(z) < \pi - \alpha\}$. Note that there is a unique $\frac{1}{2}\pi$-line joining η, η'; this is called the *axis* of η, η'.

If H is an elementary hyperbolic group it has two fixed points, η, η' say. These determine $C(\eta, \eta'; \alpha)$. On the other hand $C(\eta, \eta'; \alpha)$ is invariant under $\operatorname{con}(\Delta)_{\eta\eta'}$.

Let us introduce now the Poisson kernel,

$$P(z, \zeta) = \frac{1 - |z|^2}{|z - \zeta|^2} \quad (z \in \Delta,\ \zeta \in S).$$

The set $C(p, d) = \{z | P(z, p) > d^{-1}\}$ $(p \in S,\ d > 0)$ is a horocycle at p. It has diameter

$$\kappa(d) = 2d/(1+d).$$

The vital link between geometrical and approximation problems is given by the following sequence of lemmas. They involve only simple geometry and we relegate the proofs to the next section.

Lemma 1. *Let $x \in \Delta$, $p \in S$. There are absolute constants c, c' so that*
(i) *if $x \in C(p, d)$ then $|x - p| < c\sqrt{((1 - |x|)\,d)}$,*
(ii) *if $x \notin C(p, d)$ then $|x - p| > c'\sqrt{((1 - |x|)\,d)}$.*

LEMMA 2. *Let* $0 < \alpha < \frac{1}{2}\pi$, $x \in \Delta$, $\eta, \eta' \in S$ $(\eta \neq \eta')$. *There are constants* c, c', c'', *depending only on* α *so that*

(i) *if* $x \in C(\eta, \eta'; \alpha)$ *then*

$$\min(|\eta - x|, |\eta' - x|) < c(1-|x|) < c'|\eta - \eta'|,$$

(ii) *if* $x \notin C(\eta, \eta'; \alpha)$ *then*

$$\min(|\eta - x|, |\eta' - x|) > c'' \min((1-|x|), |\eta - \eta'|).$$

LEMMA 3. *Let* $p_1, p_2 \in S$. *There are absolute constants* c, c' *so that*

(i) *if* $C(p_1, d_1)$ *and* $C(p_2, d_2)$ *meet*

$$|p_1 - p_2| < c\sqrt{(d_1 d_2)},$$

(ii) *if* $C(p_1, d_1)$ *and* $C(p_2, d_2)$ *do not meet*

$$|p_1 - p_2| > c'\sqrt{(d_1 d_2)}.$$

LEMMA 4. *Let* $0 < \alpha < \frac{1}{2}\pi$, $\zeta, \zeta', \eta, \eta' \in S$. *There are constants* c, c' *depending only on* α *so that*

(i) *if* $C(\eta, \eta'; \alpha)$ *and* $C(\zeta, \zeta'; \alpha)$ *meet*

$$\min(|\eta - \zeta|, |\eta' - \zeta|, |\eta - \zeta'|, |\eta' - \zeta'|) < c \min(|\eta - \eta'|, |\zeta - \zeta'|),$$

(ii) *if* $C(\eta, \eta'; \alpha)$ *and* $C(\zeta, \zeta'; \alpha)$ *do not meet*

$$\min(|\eta - \zeta|, |\eta' - \zeta|, |\eta - \zeta'|, |\eta' - \zeta'|) > c' \min(|\eta - \eta'|, |\zeta - \zeta'|).$$

Moreover, as $\alpha \to 0$, $c' \to \infty$.

LEMMA 5. *Let* $0 < \alpha < \frac{1}{2}\pi$, $\eta, \eta', p \in S$. *Then there are constants* c, c' *depending only on* α *so that*

(i) *if* $C(\eta, \eta'; \alpha)$ *and* $C(p, d)$ *meet*

$$\min(|\eta - p|, |\eta' - p|) < c \min(d, \sqrt{(d|\eta - \eta'|)}),$$

(ii) *if* $C(\eta, \eta'; \alpha)$ *and* $C(p, d)$ *do not meet*

$$\min(|\eta - p|, |\eta' - p|) > c' \min(d, \sqrt{(d|\eta - \eta'|)}).$$

LEMMA 6. *Let* $\zeta, \zeta', \eta, \eta' \in S$ *be distinct. Suppose that the axes of* ζ, ζ' *and of* η, η' *meet at an angle* ϕ. *Then there are constants* c, c' *depending only on* ϕ *so that*

$$c' \min(|\eta - \eta'|, |\zeta - \zeta'|) < \min(|\eta - \zeta|, |\eta' - \zeta|, |\eta - \zeta'|, |\eta' - \zeta'|) < c \min(|\eta - \eta'|, |\zeta - \zeta'|).$$

This completes the main sequence of lemmas. We now need some basic information on how our objects interact with a Fuchsian group G. So, fix such a group, G.

As the notion will recur from now on we make the following definition. Let P be the set of parabolic vertices of G. A set of horocycles $\{C_p\}$ $(p \in P)$ is called *admissible* if $C_{g(p)} = g(C_p)$.

Proposition 2.1 affirms the existence of such sets with several extra restrictions.

If $\{C_p\}$ is an admissible set of horocycles we define d_p by $C(p, d_p) = C_p$. Proposition 2.1 asserts that we can find an admissible set of horocycles with $d_p \leqslant 1$, $p \in P$; $d_p \leqslant 1$ is equivalent to $0 \notin C(p, d_p)$.

Let $\{C'_p\}$ be another admissible set of horocycles, and $C(p, d'_p) = C'_p$. Then from the identity

$$\frac{P(z,\zeta)}{P(w,\zeta)} = \frac{P(\gamma(z), \gamma(\zeta))}{P(\gamma(w), \gamma(\zeta))} \quad (\gamma \in \operatorname{con}(\Delta)) \tag{1}$$

we deduce that

$$d_{g(p)}/d'_{g(p)} = d_p/d'_p. \tag{2}$$

There is a finite set $p_1, \ldots, p_r \in P$ so that, if $p \in P$, $p = g(p_j)$ for some p_j, and some $g \in G$. Thus, as $d_p \leqslant 1$,

$$d'_{g(pj)} = d'_{p_j} d_{g(p_j)}/d_{p_j}$$
$$\leqslant d'_{p_j}/d_{p_j} \leqslant \max_{1 \leqslant i \leqslant r} (d'_{p_i}/d_{p_i}).$$

Thus $\{d'_p | p \in P\}$ is bounded above.

Now we need some new notation. If H is a subgroup of G we form a subset of G, $G \| H$, which is a set of representatives of the cosets $\{gH | g \in G\}$ so chosen that if $g \in G\|H$, $h \in H$ then $\mu(g) \leqslant \mu(gh)$. There is some ambiguity in this choice but it will not matter.

If H is a cyclic hyperbolic or parabolic group (the only cases of interest to us) there is a unique group $\bar{H}$, $H \subset \bar{H} \subset \mathrm{con}(\Delta)$, with an isomorphism $\theta: \bar{H} \to \mathbf{R}$, $\theta(H) = \mathbf{Z}$. If H is parabolic, $\mu(g\theta^{-1}(t))$ is a positive quadratic function in $t \in \mathbf{R}$. If H is hyperbolic $\mu(g\theta^{-1}(t))$ has the form $\alpha\,\mathrm{e}^{ut} + \beta\,\mathrm{e}^{-ut} + \gamma (\alpha, \beta > 0)$. In either case the minimal value for $t \in \mathbf{Z}$ is taken on at most two integers. In other words, there are at most two elements of gH which satisfy the defining property of an element of $G\|H$.

Lemma 7. *Let $\{C_p | p \in P\}$ be an admissible set of horocycles, $C_p = C(p, d_p)$. Fix $p \in P$. Then there is $c > 0$, depending only on G and p, so that*

(i) $d_p c^{-1} \leqslant d_{g(p)} \mu(g) \quad (g \in G)$,

(ii) $d_p c \geqslant d_{g(p)} \mu(g) \quad (g \in G\|G_p)$.

Lemma 8. *Let H be a hyperbolic subgroup of G and let η, η' be its fixed points. There is $c > 0$, depending on H, so that*

(i) $c^{-1} \leqslant \mu(g)\,|g(\eta) - g(\eta')| \quad (g \in G)$,

(ii) $c \geqslant \mu(g)\,|g(\eta) - g(\eta')| \quad (g \in G\|H)$.

Proofs. The proofs are similar. We prove lemma 7 and only indicate the modifications necessary for lemma 8.

Let $\{C'_p | p \in P\}$ be a fixed admissible set of horocycles. We shall show that, for some constant c, depending only on p,

(i) $c^{-1} \leqslant d'_{g(p)} \mu(g) \quad (g \in G)$,

(ii) $c \geqslant d'_{g(p)} \mu(g) \quad (g \in G\|G_p)$,

where d'_q is defined by $C'_q = C(q, d'_q)$. This will be sufficient to prove the lemma in view of (2).

Let x be the *summit* of C'_p; i.e. the point of $\partial C'_p$ closest (hyperbolically) to 0. This always exists and is the point of $\partial C'_p$ in Δ which meets the diameter of Δ through p. Then $g(x)$ lies on $\partial C_{g(p)}$. Note that $gG_pg^{-1} = G_{g(p)}$ preserves $gC'_p = C'_{g(p)}$.

Given $y, z \in \partial C'_p$ there is $h \in G_p$ so that $[y, h(z)] \leqslant c_1$ for a suitable c_1. So also if $y^*, z^* \in \partial C'_{g(p)}$ there is $h^* \in gG_pg^{-1}$ so that $[y^*, h^*(z^*)] \leqslant c_1$. In particular, if x^* is the summit of $C'_{g(p)}$ there is $h^* \in gG_pg^{-1}$ so that $[x^*, h^*g(x)] \leqslant c_1$. Let $h = g^{-1}h^*g \in G_p$. Then $[x^*, gh(x)] \leqslant c_1$. Let $c_2 = [0, x]$.

Then
$$[0, x^*] \leqslant [0, g(x)] \qquad \text{(definition of summit)}$$
$$\leqslant [0, g(0)] + c_2 \qquad (\text{all } g \in G).$$

Also
$$[0, x^*] \geqslant [0, gh(x)] - c_1 \qquad (h \text{ as above})$$
$$\geqslant [0, gh(0)] - (c_1 + c_2).$$

These imply
$$[0, x^*] - c_2 \leqslant \min_{h \in G_p} [0, gh(0)] \leqslant [0, x^*] + (c_1 + c_2). \tag{3}$$

If $C(q, d)$ is a horocycle with summit ξ a simple calculation shows that

$$1 + \cosh [0, \xi] = (1+d)^2/2d.$$

Then
$$\mathrm{e}^{[0, \xi]} = d^{\pm 1}.$$

We have shown above that there is a constant B so that $d'_{g(p)} \leqslant B$. Hence

$$|[0, x^*] - \ln (d^{-1}_{g(p)})| \leqslant 2|\ln B|.$$

From Beardon & Nicholls (1972) we have that, for any $\gamma \in \operatorname{con}(\Delta)$

$$|[0, \gamma(0)] - \ln (\mu(\gamma))| \leqslant \ln 2.$$

From (3) and these last two inequalities

$$|\min_{h \in G_p} \ln \mu(gh) + \ln d_{g(p)}| \leqslant c_3.$$

Exponentiating $$c_4^{-1} \leqslant (\min_{h \in G_p} \mu(gh))\, d_{g(p)} \leqslant c_4.$$

As the minimum of $\mu(gh)$ $(h \in G_p)$ occurs when $gh \in G \| G_p$ this is just the statement of the lemma.

To prove lemma 8 let L be the *axis* of H; that is, the axis of η, η'. Let x be the *summit* of L; that is the mid-point of L in the Euclidean sense. Again the summit of L is the point of L hyperbolically closest to 0.

If $r = |\eta - \eta'|$ then $\frac{1}{4}r \leqslant 1 - |x| \leqslant \frac{1}{2}r$. Let x' be the summit of $g(L)$. gHg^{-1} translates L along itself with a fixed translation length. So there is $h \in gHg^{-1}$ so that $[hg(x), x'] < c_5$. Now an analogous argument to the one above completes the proof.

5. Proofs

We shall use the methods of 'transformation' geometry to prove these lemmas. The proof will consist in general of two parts. The first is the construction of a numerical invariant from the given data which will express the problem. This will involve reference to a 'canonical' situation (the 'transformation'). The second step is the deduction of the required inequalities from the invariant.

Before starting we note a few formulae. If $\gamma \in \operatorname{con}(\Delta)$ then

$$\gamma(C(p, d)) = C(\gamma(p), |\gamma'(p)|\, d), \tag{1}$$

if $z, w \in \bar{\Delta}$, $$|\gamma(z) - \gamma(w)| = |\gamma'(z)|^{\frac{1}{2}} |\gamma'(w)|^{\frac{1}{2}} |z - w|, \tag{2}$$

and if $z \in \Delta$, $$1 - |\gamma(z)|^2 = |\gamma'(z)|\, (1 - |z|^2). \tag{3}$$

These are easily checked and we shall use them repeatedly. We note also the trivial inequality, if $z \in \Delta$,

$$1 - |z| \leqslant 1 - |z|^2 \leqslant 2(1 - |z|). \tag{4}$$

Proof of lemma 4.1. For a horocycle $C = C(p, d)$ let $p(C) = p$, $d(C) = d$. They are uniquely determined by C.

Then we form, for $x \in \Delta$, $$A(C, x) = \frac{|x - p(C)|^2}{d(C)\,(1 - |x|^2)}.$$

By (1), (2), (3), if $\gamma \in \operatorname{con}(\Delta)$, $$A(\gamma(C), \gamma(x)) = A(C, x).$$

In this case $x \in C(p, d)$ means, by the definition of a horocycle, given at the beginning of § 4, that

$$\frac{|x - p|^2}{1 - |x|^2} < d(C)^{-1}.$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

So $x \in C$ if and only if $A(C, x) < 1$. In this case there is no need to find a 'canonical' situation. We can now prove lemma 4.1.

If $x \in C$ then
$$\frac{|x-p(C)|^2}{d(C)\,(1-|x|^2)} < 1.$$
With the use of (4) this shows
$$|x-p(C)| < \sqrt{[2d(C)\,(1-|x|)]},$$
which proves (i).

If $x \notin C$
$$\frac{|x-p(C)|^2}{d(C)\,(1-|x|^2)} > 1$$
so that, by (4),
$$|x-p(C)| \geqslant \sqrt{\{(1-|x|)\,d(C)\}},$$
which proves (ii).

Proof of lemma 4.2. We need another description of $C(\eta, \eta'; \alpha)$. We have already noted that $C(0, \infty; \alpha)$ is a cone of apex angle $\pi - 2\alpha$ symmetric about the imaginary axis (we are now working on $\boldsymbol{H}$). Let K be a circle, centred on a point of the imaginary axis, touching $\partial C(0, \infty; \alpha)$. K is also a hyperbolic circle. Then it is clear that if $z \in C(0, \infty; \alpha)$ there is $r \in]0, \infty[$ so that $rz \in K$ and conversely. As $z \mapsto rz$ preserves hyperbolic distances it follows that $C(0, \infty; \alpha)$ consists of all points of $\boldsymbol{H}$ which are less than some hyperbolic distance $d(\alpha)$ from the imaginary axis. Now we can return to Δ.

$C(\eta, \eta'; \alpha)$ is the set of points less than some hyperbolic distance $d(\alpha)$ from the axis of η, η'. Then $\gamma \in \operatorname{con}(\Delta)$ so that $\gamma(\eta) = 1$, $\gamma(\eta') = -1$. As α and hyperbolic distances are invariant under $\operatorname{con}(\Delta)$ it follows that $d(\alpha)$ depends on α only.

We consider the special case $\eta = 1$, $\eta' = -1$. The α-lines are
$$|z \pm \mathrm{i} \tan \alpha| = \sec \alpha.$$
From this one finds
$$\cosh d(\alpha) = \operatorname{cosec} \alpha.$$
Now form
$$A(\eta, \eta'; w) = \frac{|\eta-\eta'|\,(1-|w|^2)}{|w-\eta|\,|w-\eta'|} \quad (\eta, \eta' \in S,\ w \in \Delta).$$
By (2), (3), if $\gamma \in \operatorname{con}(\Delta)$,
$$A(\gamma(\eta), \gamma(\eta'); \gamma(w)) = A(\eta, \eta'; w).$$
If we choose γ so that $\gamma(\eta) = 1$, $\gamma(\eta') = -1$, $\gamma(w)$ is imaginary, then γ is unique. If $\gamma(w) = \mathrm{i}\xi$
$$A(\eta, \eta'; w) = 2\frac{1-\xi^2}{1+\xi^2}.$$
If $\mathrm{i}\xi$ lies on a β-line through $1, -1$ then
$$\pm \xi = \frac{1-\sin\beta}{\cos\beta}.$$
So
$$\frac{1-\xi^2}{1+\xi^2} = \sin\beta.$$

Hence w lies in $C(\eta, \eta'; \alpha)$ if and only if
$$A(\eta, \eta'; w) > 2\sin\alpha.$$

Now we can deduce the lemma directly. We may suppose that $|x-\eta| \leqslant |x-\eta'|$ without loss of generality. As $\eta, \eta' \in S$
$$|x-\eta'| \geqslant |x-\eta| \geqslant (1-|x|). \tag{5}$$
By the triangle inequality
$$|\eta-\eta'| \leqslant |x-\eta| + |x-\eta'| \leqslant 2|x-\eta'|. \tag{6}$$

Suppose now that $x \in C(\eta, \eta'; \alpha)$. Then

$$|\eta - \eta'|\,(1-|x|^2) > 2\sin\alpha\,|x-\eta|\,|x-\eta'|.$$

By (6) $$(1-|x|^2) > \sin\alpha\,|x-\eta| \tag{7}$$

and so, by (4), $$2\operatorname{cosec}\alpha(1-|x|) > |x-\eta|. \tag{8}$$

From (7) and (5) $$|\eta-\eta'|\,(1-|x|^2) > 2\sin\alpha(1-|x|)^2.$$

So, from (4) $$(1-|x|) < \operatorname{cosec}\alpha|\eta-\eta'|.$$

This and (8) constitute (i).

Now suppose $x \notin C(\eta, \eta'; \alpha)$. Then

$$|\eta-\eta'|\,(1-|x|^2) \leqslant 2\sin\alpha\,|x-\eta|\,|x-\eta'|.$$

The inequality $$|x-\eta'| \leqslant |x-\eta| + |\eta-\eta'|$$

gives $$|\eta-\eta'|\,(1-|x|^2) \leqslant 2\sin\alpha\,|x-\eta|\,(|x-\eta|+|\eta-\eta'|).$$

If $$|x-\eta| < |\eta-\eta'|$$

we obtain $$1-|x|^2 \leqslant 4\sin\alpha\,|x-\eta|.$$

Thus, by using (4), $$|x-\eta| \geqslant \min\,(|\eta-\eta'|,\ \tfrac{1}{4}(1-|x|)/\sin\alpha),$$

which proves (ii).

It should be observed that the greatest labour expended in this proof was in deriving the analytic expression defining $C(\eta, \eta'; \alpha)$.

Proof of lemma 4.3. Let C, C' be two horocycles. Define

$$A(C, C') = \frac{|p(C)-p(C')|^2}{d(C)\,d(C')}.$$

Again, if $\gamma \in \operatorname{con}(\Delta)$, $A(\gamma C, \gamma C') = A(C, C')$ by (1) and (2). We can assume $p(C) \neq p(C')$ as our results will be trivial otherwise. Then it is easy to see that there is γ so that

$$\gamma(p(C)) = 1, \quad \gamma(p(C')) = -1 \quad \text{and} \quad \gamma(C) = \{z|\ |z-\tfrac{1}{2}| < \tfrac{1}{2}\}.$$

Then, without any trouble one finds that C, C' intersect if and only if

$$A(C, C') \leqslant 4.$$

If $C = C(p_1, d_1)$ and $C' = C(p_2, d_2)$ meet

$$|p_1 - p_2|^2 \leqslant 4d_1 d_2$$

or $$|p_1-p_2| \leqslant 2\sqrt{(d_1 d_2)}$$

which proves (i).

If C, C' do not intersect $$|p_1-p_2|^2 > 4d_1 d_2$$

which proves (ii).

It is convenient to prove lemmas 4.4 and 4.6 together.

Proof of lemmas 4.4 *and* 4.6. First let us reduce lemma 4.4 somewhat. We have shown, in the proof of lemma 4.2, that $C(\eta, \eta'; \alpha)$ consists of all points hyperbolically closer than $d(\alpha)$ to the axis of η, η'. Let L (resp. M) be the axis of η, η' (resp. ζ, ζ'). If $C(\zeta, \zeta'; \alpha)$ and $C(\eta, \eta'; \alpha)$ intersect there is $w \in \Delta$ closer than $d(\alpha)$ to both L, M. Thus the (hyperbolic) distance between L and M is less than $2d(\alpha)$. We shall show that the converse is true. Let l be the distance between L and M where we assume $l < 2d(\alpha)$. There is a geodesic K from a point of L to a point of M of length k, where $l \leqslant k < 2d(\alpha)$. Let u be (hyperbolic) mid-point of K. Then the distance of u to L(resp. M) is $\leqslant \frac{1}{2}k < d(\alpha)$. Thus u lies in both $C(\eta, \eta'; \alpha)$ and $C(\zeta, \zeta'; \alpha)$. Thus $C(\eta, \eta'; \alpha)$ and $C(\zeta, \zeta'; \alpha)$ intersect.

The configuration of both lemmas 4.4 and 4.6 can be described by the two pairs of points (η, η'), (ζ, ζ'). Such pairs of pairs are classified, up to the action of con (Δ), by the familiar cross-ratio

$$A(\zeta, \zeta'; \eta, \eta') = \frac{(\zeta-\eta')(\zeta'-\eta)}{(\zeta-\eta)(\zeta'-\eta')}.$$

This, as is well known, is a real number (if $\zeta, \zeta', \eta, \eta' \in S$). L and M intersect in Δ if and only if $A(\zeta, \zeta'; \eta, \eta') < 0$. Furthermore, if L and M intersect at an angle ϕ

$$A(\zeta, \zeta'; \eta, \eta') = -\cot^2(\tfrac{1}{2}\phi).$$

If $A > 0$ $A(\zeta, \zeta'; \eta, \eta')$ depends only on the ordering of ζ, ζ' and of η, η' and on the hyperbolic distance between L and M.

The invariant A is not very suitable for our purposes. We define

$$B(\zeta, \zeta'; \eta, \eta') = \frac{|\zeta-\eta|\,|\zeta-\eta'|\,|\zeta'-\eta|\,|\zeta'-\eta'|}{|\zeta-\zeta'|^2\,|\eta-\eta'|^2}.$$

If L and M intersect at an angle ϕ then we find that

$$B(\zeta, \zeta'; \eta, \eta') = \tfrac{1}{4}\sin^2\phi.$$

We split the proof of lemma 4.4 into two cases depending on whether L and M intersect. Assume, as we can, that

$$|\zeta-\eta| = \min\left(|\zeta-\eta|, |\zeta'-\eta|, |\zeta-\eta'|, |\zeta'-\eta'|\right)$$
$$|\eta-\eta'| \leqslant |\zeta-\zeta'|.$$

Now we can prove lemma 4.4 in the case that L and M intersect. With this normalization, from the triangle inequality we obtain the following inequalities.

As $$|\eta-\zeta'| + |\zeta-\eta| \geqslant |\zeta-\zeta'|,$$

hence $$|\eta-\zeta'| \geqslant \tfrac{1}{2}|\zeta-\zeta'|. \tag{9}$$

Likewise $$|\eta'-\zeta| \geqslant \tfrac{1}{2}|\eta-\eta'|. \tag{10}$$

Also $$|\zeta'-\eta'| + |\eta'-\eta| + |\eta-\zeta| \geqslant |\zeta-\zeta'|,$$

so $$2|\zeta'-\eta'| + |\eta-\eta'| \geqslant |\zeta-\zeta'|. \tag{11}$$

In the case of lemma 4.4 only case (i) arises and

$$B(\zeta, \zeta'; \eta, \eta') = \tfrac{1}{4}\sin^2\phi \leqslant \tfrac{1}{4}.$$

So $$|\zeta-\eta|\,|\zeta'-\eta|\,|\zeta-\eta'|\,|\zeta'-\eta'| \leqslant \tfrac{1}{4}|\zeta-\zeta'|^2\,|\eta-\eta'|^2.$$

From (9) and (10) we obtain $$|\zeta-\eta|\,|\zeta'-\eta'| \leqslant |\zeta-\zeta'|\,|\eta-\eta'|.$$

Now we split cases. If $|\eta-\eta'| \leqslant \frac{1}{2}|\zeta-\zeta'|$ then (11) gives

$$2|\zeta'-\eta'| \geqslant \tfrac{1}{2}|\zeta-\zeta'|$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

and we obtain $$|\zeta-\eta| \leqslant 4\,|\eta-\eta'|.$$
If $$|\eta-\eta'| \geqslant \tfrac{1}{2}|\zeta-\zeta'|,$$
then, as $$|\zeta-\eta| \leqslant |\zeta'-\eta'|$$
one has $$|\zeta-\eta| \leqslant \sqrt{2}\,|\eta-\eta'|.$$
In either case $$|\zeta-\eta| \leqslant 4\,|\eta-\eta'|.$$
This is the assertion of the lemma.

In the case of lemma 4.6 we make the same normalization. The same argument as above shows
$$|\zeta-\eta| \leqslant \max\,(4\sin^2\phi,\ \sqrt{2}\,|\sin\phi|)\,|\eta-\eta'|. \tag{12}$$
On the other hand from
$$|\zeta-\eta|\,|\zeta-\eta'|\,|\zeta'-\eta|\,|\zeta'-\eta'| = \tfrac{1}{4}\sin^2\phi\,|\zeta-\zeta'|^2\,|\eta-\eta'|^2$$
we obtain, by the triangle inequality
$$|\zeta-\eta|\,(|\zeta-\eta|+|\eta-\eta'|)\,(|\zeta-\eta|+|\zeta-\zeta'|)\,(|\zeta-\eta|+|\zeta-\zeta'|+|\eta-\eta'|) \geqslant \tfrac{1}{4}\sin^2\phi\,|\zeta-\zeta'|^2\,|\eta-\eta'|^2.$$
If $|\zeta-\eta| < \theta|\eta-\eta'|$ we have, on recalling that $|\eta-\eta'| \leqslant |\zeta-\zeta'|$ the upper bound
$$\theta(\theta+1)^2\,(\theta+2)\,|\zeta-\zeta'|^2\,|\eta-\eta'|^2$$
for the left hand side. As $\theta(\theta+1)^2\,(\theta+2)$ is increasing we can find $\theta_0 > 0$ so that, if $\theta \leqslant \theta_0$
$$\theta(\theta+1)^2\,(\theta+2) \leqslant \tfrac{1}{4}\sin^2\phi.$$
For example, we could take $\theta_0 = \frac{1}{48}\sin^2\phi$.

Hence, if $|\zeta-\eta| < \theta_0\,|\eta-\eta'|$, we would have a contradiction. Thus
$$|\zeta-\eta| \geqslant \theta_0\,|\eta-\eta'| \tag{13}$$
(12) and (13) are the assertions of lemma 4.6.

Now consider the case $A > 0$. Again we form the invariant B. We can find $\gamma \in \mathrm{con}\,(\Delta)$ so that, for some $\theta \in]0, \pi[$
$$\{\gamma\eta, \gamma\eta'\} = \{+1, -1\}, \{\gamma\zeta, \gamma\zeta'\} = \{e^{i\theta}, -e^{-i\theta}\}.$$
It follows easily that B depends only on the hyperbolic distance between L and M. If this distance is D then we find after an easy calculation that
$$\cosh^2 D = 4B(\zeta, \zeta'; \eta, \eta') + 1.$$

We can now prove lemma 4.4 completely. Assume (i), which in view of our earlier discussion, means $D < 2d(\alpha)$. Thus
$$B(\zeta, \zeta'; \eta, \eta') \leqslant \tfrac{1}{4}(\cosh^2 2d(\alpha) - 1)$$
and the conclusion follows exactly the deduction of the conclusion in the case when L and M intersect.

Assume $C(\zeta, \zeta; \alpha)$ and $C(\eta, \eta'; \alpha)$ do not meet. Then $D \geqslant 2d(\alpha)$ and hence
$$B(\zeta, \zeta'; \eta, \eta') \geqslant \tfrac{1}{4}(\cosh^2 2d(\alpha) - 1)$$
and we can deduce the conclusion just as we deduced (13) above. We leave the details as they present no problems.

This leaves only lemma 4.5.

Proof of lemma 4.5. This follows the well-worn pattern. We are given as data η, η' and a horocycle C. C and $C(\eta, \eta'; \alpha)$ intersect if and only if the distance between the axis L of η, η' and C is less than $d(\alpha)$. Let

$$A(\eta, \eta'; C) = \frac{|\eta - p(C)|\,|\eta' - p(C)|}{|\eta - \eta'|\, d(C)}.$$

By (1) and (2) this satisfies, for $\gamma \in \mathrm{con}\,(\Delta)$,

$$A(\gamma(\eta), \gamma(\eta'); \gamma(C)) = A(\eta, \eta'; C).$$

We can find a 'canonical' form by choosing γ so that

$$\gamma(\eta) = 1, \quad \gamma(\eta') = -1, \quad \gamma(p(C)) = \pm \mathrm{i}.$$

As γ is determined by its action on three points it is uniquely determined. A straightforward calculation shows that C and $C(\eta, \eta'; \alpha)$ meet if and only if

$$A(\eta, \eta'; C) < \cot\left(\tfrac{1}{2}\alpha\right).$$

We let $C = C(p, d)$. Without loss of generality we may suppose that $|p - \eta| \leqslant |p - \eta'|$. So by the triangle inequality

$$|\eta - \eta'| \leqslant |p - \eta| + |p - \eta'| \leqslant 2|p - \eta'|. \tag{14}$$

Now assume that $C(p, d)$ and $C(\eta, \eta'; \alpha)$ meet

$$|p - \eta|\,|p - \eta'| \leqslant \cot\left(\tfrac{1}{2}\alpha\right) d\,|\eta - \eta'|.$$

So, clearly, as $|p - \eta| \leqslant |p - \eta'|$,

$$|p - \eta|^2 \leqslant \cot\left(\tfrac{1}{2}\alpha\right) d\,|\eta - \eta'|. \tag{15}$$

But using (14) instead gives

$$|p - \eta| \leqslant 2\cot\left(\tfrac{1}{2}\alpha\right) d, \tag{16}$$

(15) and (16) constitute the conclusion of lemma 4.5. (i).

Suppose now that $C(p, d)$ and $C(\eta, \eta'; \alpha)$ do not meet. Then

$$|p - \eta|\,|p - \eta'| \geqslant \cot\left(\tfrac{1}{2}\alpha\right) d\,|\eta - \eta'|.$$

Hence, as

$$|p - \eta'| \geqslant |\eta - \eta'| + |p - \eta|,$$

$$|p - \eta|\,(|p - \eta| + |\eta - \eta'|) \geqslant \cot\left(\tfrac{1}{2}\alpha\right) d\,|\eta - \eta'|.$$

Suppose that $|\eta - p| \leqslant |\eta - \eta'|$. Then this gives

$$2\,|p - \eta| \geqslant \cot\left(\tfrac{1}{2}\alpha\right) d. \tag{17}$$

On the other hand, if $|\eta - p| \geqslant |\eta - \eta'|$ it gives

$$2\,|p - \eta|^2 \geqslant \cot\left(\tfrac{1}{2}\alpha\right) d\,|\eta - \eta'|. \tag{18}$$

Hence either (17) or (18) is true. This is the assertion of lemma 4.5 (ii).

6. Applications to the fixed points

We can now apply the results of § 5 in a preliminary fashion. The methods of this section are the background for those of the next. The investigation here is a development of the work of Rankin (1957).

Four essentially different cases arise and to avoid a monster theorem we have to split our results into four theorems. This multiplicity, although displeasing, seems at the present time to be unavoidable. For this section we fix a Fuchsian group G.

Theorem 1. *Let p, q be parabolic vertices of G. There is a constant $c > 0$ so that, if $g(p) \neq q$,*

$$|g(p)-q| > c/\sqrt{(\mu(g))}. \tag{1}$$

Moreover, if for some $k > 0$

$$0 < |g_n(p)-q| \leqslant k/\sqrt{(\mu(g_n))} \tag{2}$$

for a sequence (g_n) in G then the g_n belong to at most a finite number of right cosets of G_q.

Proof. By proposition 2.1 we can find an admissible set $\{C_p\}$ of horocycles which are disjoint. As $g(C_p)$ and C_q are disjoint lemma 4.3 (ii) gives

$$|g(p)-q| > c_1\sqrt{\{d(C_{g(p)})\,d(C_q)\}}.$$

By lemma 4.7 (i)

$$d(C_{g(p)}) \geqslant c_2\,\mu(g)^{-1}. \tag{3}$$

These two inequalities give the first assertion.

We prove the second. We claim that we can find d so that $C(q,d)$ meets all $g_n(C_p)$. For if $g_n(C_p)$ and $C(q,d)$ are disjoint by lemma 4.3 (ii)

$$|g_n(p)-q| \geqslant c_1\sqrt{\{d(C_{g_n(p)})\,d\}}.$$

By lemma 4.7 (i)

$$d(C_{g_n(p)}) \geqslant c_2\,\mu(g_n)^{-1}.$$

Thus

$$|g_n(p)-q| \geqslant c_1 c_2^{\frac{1}{2}}\, d^{\frac{1}{2}}/\mu(g_n)^{\frac{1}{2}}.$$

If $d > k^2/(c_1^2 c_2)$ this contradicts (2). Hence if this is so $g_n(C_p)$ and $C(q,d)$ meet. So $g_n(C_p)$ and $\partial C(q,d)$ meet. However there is a compact subset K of $\partial C(q,d)$ so that

$$G_q K = \partial C(q,d).$$

Thus for g_n there is $h_n \in G_q$ so that $h_n g_n(C_p)$ meets K. By lemma 4.7 (ii) only a finite subset of an admissible set of horocycles has diameter greater than a given quantity. Thus there is a finite subset B of G so that, given n, there is $b_n \in B$ so that

$$h_n g_n(C_p) = b_n(C_p).$$

So, given n there is $k_n \in G_p$

$$g_n = h_n^{-1} b_n k_n.$$

As $g_n(p) = h_n^{-1} b_n(p)$ we have from (1)

$$|g_n(p)-q| > c/\sqrt{\{\mu(h_n^{-1} b_n)\}}.$$

On combining this with (2), we find

$$\mu(h_n^{-1} b_n k_n) \leqslant c_3\,\mu(h_n^{-1} b_n).$$

Thus, by using $\mu(\gamma_1\gamma_2) \leqslant \mu(\gamma_1)\,\mu(\gamma_2)$ $(\gamma_1, \gamma_2 \in \mathrm{con}\,(\Delta))$

$$\mu((b_n^{-1} h_n b_n)^{-1} k_n) \leqslant \{\mu(b_n)^2 c_3\}\,\mu((b_n^{-1} h_n b_n)^{-1}). \tag{4}$$

As the b_n run through a finite set, without loss of generality we can take b_n constant and equal to b say. $b^{-1} h_n b \in G_{b^{-1}(q)}$. If $b^{-1}(q) = p$ then $g_n(p) = q$. So $b^{-1}(q) \neq p$. We need the lemma

Lemma 1. *If H_1, H_2 are parabolic subgroups of* con (Δ) *with distinct fixed points p_1, p_2 there is a constant c_4 so that, if $h_1 \in H_1$, $h_2 \in H_2$*

$$\mu(h_1 h_2) \geqslant c_4\,\mu(h_1)\,\mu(h_2).$$

This applied to the case $H_1 = b^{-1} G_q b$, $H_2 = G_q$, gives a constant c_5 depending on b, p, q so that

$$\mu((b^{-1} h_n b)^{-1} k_n) \geqslant c_5\,\mu((b^{-1} h_n b)^{-1})\,\mu(k_n).$$

With (4) this implies that

$$\mu(k_n) \leqslant c_6.$$

So the set of possible k_n is finite. Thus
$$\{g_n = h_n^{-1} b_n k_n^{-1}\}$$
runs through at most a finite number of right cosets of G_q.

It only remains to prove the lemma.

Proof of lemma. There is $\gamma \in \text{con}\,(\Delta)$ so that
$$\gamma(p_1) = 1, \quad \gamma(p_2) = -1.$$
As, for $g \in \text{con}\,(\Delta)$,
$$\mu(\gamma)^{-2}\mu(\gamma g \gamma^{-1}) \leqslant \mu(g) \leqslant \mu(\gamma)^2 \mu(\gamma g \gamma^{-1}),$$
we need only prove the lemma for $p_1 = 1$, $p_2 = -1$. If g_1 (resp. g_2) is a parabolic element fixing 1 (resp. -1) it has the form
$$g_1 = \begin{pmatrix} 1+it & -it \\ it & 1-it \end{pmatrix} \quad \left(\text{resp. } g_2 = \begin{pmatrix} 1+iu & iu \\ -iu & 1-iu \end{pmatrix}\right).$$
Easy calculations show that $\mu(g_1) = 4t^2+2$, $\mu(g_2) = 4u^2+2$ and
$$\mu(g_1 g_2) = 2(1+2(u-t)^2+8u^2t^2).$$
So
$$\mu(g_1 g_2) \geqslant 2(1+8u^2t^2).$$
As
$$(1+8u^2t^2) \geqslant 9^{-1}(1+2u^2)(1+2t^2)$$
the lemma follows at once.

Theorem 1 is typical of those to follow.

THEOREM 2 (Rankin 1957). *Let p be a parabolic vertex and η, η' a pair of conjugate hyperbolic fixed points. There is a constant $c > 0$ so that, for $g \in G$*
$$|g(p)-\eta| > c/\mu(g). \tag{5}$$
Moreover, if for some $k > 0$
$$|g_n(p)-\eta| \leqslant k/\mu(g_n) \tag{6}$$
for a sequence (g_n) in G then the g_n belong to only a finite number of right cosets of $G_{\eta\eta'}$.

Proof. Given G, I claim that there is α $(0 < \alpha < \frac{1}{2}\pi)$ and an admissible set of horocycles $\{C_q\}$ so that no C_q meets $C(\eta, \eta'; \alpha)$. For $G_{\eta\eta'}$ acts on $C(\eta, \eta'; \alpha)$ and has a relatively compact fundamental domain K there. If C_q meets γK $(\gamma \in G_{\eta\eta'})$ then $C_{\gamma^{-1}(q)}$ meets K. So we have only to ensure that no C_q meets K. As K is relatively compact this can be done in view of lemma 4.7 (ii).

Hence $g(C_p)$ does not meet $C(\eta, \eta'; \alpha)$. By lemma 4.5 (ii)
$$|g(p)-\eta| \geqslant c_1 \min\,(d(C_{g(p)}), d(C_{g(p)})^{\frac{1}{2}}\,|\eta-\eta'|^{\frac{1}{2}}).$$
As $\{d(C_q)\}$ is bounded above one has
$$|g(p)-\eta| \geqslant c_2\, d(C_{g(p)}).$$
By lemma 4.7(ii), $d(C_{g(p)}) \geqslant c_3/\mu(g)$, where c_3 depends on p but not on g. Hence
$$|g(p)-\eta| \geqslant c_4/\mu(g),$$
which proves the first part.

Suppose now we are given a sequence (g_n). We claim that there is d so that $g_n(C(p,d))$ and $C(\eta, \eta'; \frac{1}{4}\pi)$ meet. For, if they do not, by lemma 4.5 (ii)
$$|g_n(p)-\eta| \geqslant c_5 \min\,(d(C_{g_n(p)}), (d(C_{g_n(p)})\,|\eta-\eta'|)^{\frac{1}{2}}).$$
By lemma 4.7 (i) this exceeds
$$c_6 \min\,(\mu(g_n)^{-1} d, \mu(g_n)^{-\frac{1}{2}} d^{\frac{1}{2}}\,|\eta-\eta'|^{\frac{1}{2}}).$$

If we choose $d > \max(c_6^{-1}k, c_6^{-2}|\eta-\eta'|^{-1}k^2)$ this shows that

$$|g_n(p)-\eta| > k/\mu(g_n),$$

which contradicts (6). Thus for such d, $g_n(C(p,d))$ and $C(\eta,\eta';\tfrac{1}{4}\pi)$ meet. Hence, as we have seen at the beginning at the proof of this theorem, there is a compact set K so that, for some $h_n \in G_{\eta\eta'}$, $h_n^{-1}g_n(C(p,d))$ and K intersect. Also, as we have seen above only a finite number of horocycles $g(C(p,d))$ $(g \in G)$ have diameter greater than a given quantity c_7. Hence there is a finite subset B of G so that, for each n there is $b_n \in B$ so that

$$h_n^{-1}g_n(C(p,d)) = b_n(C(p,d)).$$

Thus, for each n there is $k_n \in G_p$, $h_n \in G_{\eta\eta'}$ and $b_n \in B$ so that

$$g_n = h_n b_n k_n.$$

Thus we can finish the proof of the theorem along analogous lines to theorem 1 if we have the lemma:

Lemma 2. *If H_1 is a hyperbolic subgroup of* con (Δ) *and H_2 is a parabolic subgroup of* con (Δ) *whose fixed point is not one of the fixed points of H_1. Then there is a constant $c_8 > 0$ so that, for $h_1 \in H_1$, $h_2 \in H_2$*

$$\mu(h_1 h_2) \geqslant c_8\mu(h_1)\mu(h_2), \quad \mu(h_2 h_1) \geqslant c_8\mu(h_1)\mu(h_2).$$

As before it is easy to reduce this to a special case; say when the fixed points of H_1 are ± 1 and that of H_2 is i. The lemma is proved then by direct calculation in the same way as lemma 1. As there is nothing to be gained from the proof we omit it.

The proof of theorem 2 proceeds unhindered along the pattern set in the proof of theorem 1 provided we recall that no point is both a hyperbolic and a parabolic fixed point.

We shall state another theorem, without proof, which can be proved by analogous methods.

Theorem 3. *Let η, η' be a pair of conjugate hyperbolic fixed points and p a parabolic fixed point of G. Then there is a constant $c_9 > 0$ so that*

$$|g(\eta)-p| > c_9/\sqrt{(\mu(g))}.$$

Moreover, if for some $k > 0$

$$|g_n(\eta)-p| \leqslant k/\sqrt{(\mu(g_n))}$$

for a sequence (g_n) in G then the g_n belong to at most a finite number of right cosets of G_p.

Finally we state

Theorem 4. *Let η, η' and ζ, ζ' be two pairs of hyperbolic fixed points of G. Then there is a constant $c_{10} > 0$ so that if $g(\eta) \neq \zeta$*

$$|g(\eta)-\zeta| > c_{10}/\mu(g).$$

Moreover, if for some $k > 0$ there is a sequence (g_n) in G so that

$$|g_n(\eta)-\zeta| \leqslant k/\mu(g_n)$$

then the g_n belong to at most a finite number of right cosets of $G_{\eta\eta'}$.

The proof of this is somewhat different to the previous ones and so we indicate the modifications necessary.

Proof. Let L (resp. M) be the axis of η, η' (resp. ζ, ζ'). Let us look at the set

$$T = \{g \in G \mid gL \text{ meets } M \text{ in } \Delta\}$$

$G_{\zeta\zeta'}$ acts on M and there is a compact interval $I \subseteq M$ so that $G_{\zeta\zeta'}I = M$. So if $g \in T$ there is $\gamma \in G_{\zeta\zeta'}$ so that $\gamma g(L)$ meets M in I.

I is compact in Δ and so, as $\gamma g(L)$ is the axis of $\gamma g(\eta)$, $\gamma g(\eta')$ there is c_{11} so that

$$|\gamma g(\eta) - \gamma g(\eta')| \geqslant c_{11}.$$

From lemma 4.8 (ii) it follows there is $h \in G_{\eta\eta'}$ so that $\mu(\gamma g h) \leqslant c_{12}$ for some c_{12}. Hence there is a finite set V_0 in G so that

$$T \subseteq G_{\zeta\zeta'} V_0 G_{\eta\eta'}.$$

From the definition of T it is made up of a number of double cosets in $G_{\zeta\zeta'}\backslash G/G_{\eta\eta'}$. Hence there is a finite set V in G so that

$$T = G_{\zeta\zeta'} V G_{\eta\eta'}.$$

Let $\nu \in V$. Then let $\phi(\nu)$ be the angle at which νL and M intersect. As all our maps are conformal this is also the angle at which $\gamma\nu h(L)$ and M intersect if

$$\gamma \in G_{\zeta\zeta'}, \quad h \in G_{\eta\eta'}.$$

Let us consider g for which $g(L)$ and M do not meet. As a point cannot be a fixed point of two distinct hyperbolic subgroups, $g(L)$ and M cannot be asymptotic. Hence the (hyperbolic) distance between them is positive. Let

$$U(D) = \{g \in G \,|\, [g(L), M] < D\}.$$

We shall show that $U(D)$ consists of a finite number of double cosets of $G_{\zeta\zeta'}\backslash G/G_{\eta\eta'}$. Clearly it is a union of such cosets.

There is a compact interval $I \subseteq M$ so that $G_{\zeta\zeta'} I = M$. Suppose the closest distance between gL and M is achieved by y on $g(L)$ and z on M. y, z are then known to be unique. There is $\gamma \in G_{\zeta\zeta'}$ so that $\gamma(z) \in I$.

Thus some point of $\gamma g(L)$ is within D of a compact set; that is, some point of $\gamma g(L)$ is in a compact set K (which depends on D). By lemma 4.8 (ii) we now deduce that there is $h \in G_{\eta\eta'}$ so that

$$\mu(\gamma g h) \leqslant c_{13}(D).$$

Hence there is a finite set $W(D)$ so that

$$U(D) = G_{\zeta\zeta'} W(D) G_{\eta\eta'}.$$

In particular the distance between gL and M, if non-zero, is bounded below. Hence there is α so that, if gL and M do not intersect, $gC(\eta, \eta'; \alpha)$ and $C(\zeta, \zeta'; \alpha)$ do not meet.

Let $U = \bigcup_{D>0} U(D)$. Then G is the disjoint union of T and U. If we split cases as g is in T or U and use either lemma 4.6 (ii) or 4.4 (ii) (and 4.8) we can complete the argument exactly along the lines of theorem 1. We need the lemma

Lemma 3. *Let H_1, H_2 be two hyperbolic subgroups of* con (Δ) *without a common fixed point. Then there is $c_{14} > 0$ so that, if $h_1 \in H_1$, $h_2 \in H_2$*

$$\mu(h_1 h_2) \geqslant c_{14}\, \mu(h_1)\, \mu(h_2).$$

Proof. As in the proof of lemma 1 we may normalize by conjugating H_1 and H_2. So we may suppose that H_1 fixes 1, -1 and H_2 is arbitrary.

An arbitrary element h_1 of H_1 has the form

$$h_1 = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

for some $t \in \mathbf{R}$.

To express an arbitrary element of H_2 we require two parameters, $u \in \boldsymbol{R}$, $\nu \in \boldsymbol{C}$ with $1+u^2 = |\nu|^2$ which depend only on H_2. Then an arbitrary element of H_2 is given by

$$h_2 = \begin{pmatrix} \cosh s + \mathrm{i}u \sinh s & \nu \sinh s \\ \bar{\nu} \sinh s & \cosh s - \mathrm{i}u \sinh s \end{pmatrix}$$

for some $s \in \boldsymbol{R}$. The fixed points are $(1+\mathrm{i}u)/\bar{\nu}$ and $-(1-\mathrm{i}u)/\bar{\nu}$.

The fixed points of H_2 are not ± 1 if

$$1 \pm \mathrm{i}u \pm \nu \neq 0. \tag{7}$$

An easy calculation shows

$$\mu(h_1 h_2) = \tfrac{1}{2}\{|(1+\mathrm{i}u+\bar{\nu})\,\mathrm{e}^{s+t} + (1-\mathrm{i}u+\bar{\nu})\,\mathrm{e}^{-(s+t)} + (1+\mathrm{i}u-\bar{\nu})\,\mathrm{e}^{s-t} + (1-\mathrm{i}u-\bar{\nu})\,\mathrm{e}^{t-s}|^2$$
$$+ |(1-\mathrm{i}u+\nu)\,\mathrm{e}^{s+t} + (-1+\mathrm{i}u-\nu)\,\mathrm{e}^{-(s+t)} + (-1+\mathrm{i}u+\nu)\,\mathrm{e}^{s-t} + (1+\mathrm{i}u-\nu)\,\mathrm{e}^{t-s}|^2\}.$$

From this and (7) it is clear that for $|s| \geqslant s_0 |t| \geqslant t_0$

$$\mathrm{e}^{-2(|s|+|t|)}\mu(h_1 h_2) \geqslant c_{15}.$$

On the other hand, $\mu(h_1) \leqslant c_{16}\,\mathrm{e}^{2|s|}, \quad \mu(h_2) \leqslant c_{17}\,\mathrm{e}^{2|t|}.$

The last three inequalities imply the assertion of the lemma if $\mu(h_1), \mu(h_2) \geqslant c_{18}$.

If either $\mu(h_1)$ (or $\mu(h_2)$) is less than c_{18} we have

$$\mu(h_1 h_2) \geqslant \mu(h_1^{-1})^{-1}\mu(h_2)$$
$$\geqslant c_{18}^{-1}\mu(h_2).$$

Thus if $\mu(h_1) \leqslant c_{18}$ we have $\mu(h_1 h_2) \geqslant c_{18}^{-2}\mu(h_2)\,\mu(h_1).$

This proves the lemma in all cases.

The results of this chapter give a complete description of how the fixed points of a Fuchsian group approximate one another. They will also, in the sequel, give examples (especially of 'worst possible' type of behaviour).

7. Uniform approximation

We are now in a position to state and prove the major theorems in this area. The theorems about to be presented generalize Dirichlet's theorem in the theory of diophantine approximation. These theorems are our major technical tool in the study of groups of the first kind.

Theorem 1. *Let G be a Fuchsian group containing parabolic elements and let $P_0 = \{p_1, \ldots, p_s\}$ be a complete set of parabolic vertices inequivalent under G. There is a constant $c > 0$ with the following property: if $x \in L_G$, $X \geqslant 2$, there is $p \in P_0$ and $g \in G$ with $\mu(g) \leqslant X$ so that*

$$|g(p) - x| \leqslant c/\surd(\mu(g)\,X). \tag{1}$$

Moreover, there is a constant $c' > 0$ so that if $q_i \in P_0$ $(i = 1, 2)$ then, if $g_1(q_1) \neq g_2(q_2)$,

$$|g_1(q_1) - g_2(q_2)| > c'/\surd(\mu(g_1)\,\mu(g_2)). \tag{2}$$

Theorem 2. *Let G be a Fuchsian group without parabolic elements. Let η, η' be a conjugate pair of hyperbolic fixed points. Then there is $c > 0$ with the following property: if $x \in L_G$, $X \geqslant 2$ there is $\zeta \in \{\eta, \eta'\}$, $g \in G$ with $\mu(g) \leqslant X$ so that*

$$|g(\zeta) - x| \leqslant c/X. \tag{3}$$

Moreover, there is $c' > 0$ so that, if $g_j \in G$ with $\mu(g_j) \leqslant X$, $\zeta_j \in \{\eta, \eta'\}$ $(j = 1, 2)$ then, if $g_1(\zeta_1) \neq g_2(\zeta_2)$,

$$|g_1(\zeta_1) - g_2(\zeta_2)| > c'/X. \tag{4}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Theorem 2 is false if G has parabolic elements.

These two theorems should be compared to Hedlund's lemma (theorem 3.1). The important difference is that the sizes of the elements of G concerned are completely under control. It is also possible to regard these as giving analogues to the *Farey series*; for instance, they show the connection between the classical 'circle method' and the variant used by Lehner (1964).

Another interpretation is that they describe economic covers of L_G; covers, in fact, of bounded 'depth' and whose sizes are well under control. The usefulness of these two theorems will be demonstrated in the next three sections. The remainder of this section is devoted to the proofs.

Proofs of the theorems. The 'Moreover, ...' clauses are relatively trivial and we deal with them first. The proofs are variants of the proofs of theorems 6.1 and 6.4.

In the case of theorem 1 we can find, by proposition 2.1, and admissible set of horocycles $\{C_p\}$ so that, if $p \neq q$, C_p does not meet C_q. As $g_1(C_{p_1})$ and $g_2(C_{p_2})$ do not meet lemma 4.3 (ii) gives

$$|g_1(p_1) - g_2(p_2)| \geqslant c_1(d(C_{g_1(p_1)})\, d(C_{g_2(p_2)}))^{\frac{1}{2}}.$$

By lemma 4.7 (ii), for $j = 1, 2$, $\qquad d(C_{g_j(p_j)}) \geqslant c_2\, d(C_{p_j})/\mu(g_j)$.

As the p_j belong to a finite set $\{d(C_{pj})\}$ is bounded below. On combining these remarks one obtains (2).

The case of theorem 2 is analogous. We split cases. If the axes of $g_1(\eta)$, $g_1(\eta')$ and of $g_2(\eta)$, $g_2(\eta')$ meet they do so, by the analysis in the proof of theorem 6.4 at one of a finite number of angles. In this case the conclusion follows at once from lemmas 4.6 and 4.8. Otherwise, as in the proof of theorem 6.4 there is $\alpha > 0$ so that if the axes of $g_1(\eta)$, $g_1(\eta')$ and of $g_2(\eta)$, $g_2(\eta')$ do not meet then $g_1(C(\eta, \eta'; \alpha))$ and $g_2(C(\eta, \eta'; \alpha))$ do not meet. Then the conclusion follows from lemmas 4.4 and 4.8.

We now turn to the proof of the main part of theorem 1. Recall that in § 2 we constructed regions $\Lambda_j(\alpha)$. If G is non-elementary the set

$$K_G(\alpha) = \Delta \smallsetminus \bigcup \Lambda_j(\alpha)$$

is non-empty if $\alpha \leqslant \frac{1}{2}\pi$, because there is no way of covering Δ by more than two disjoint $\frac{1}{2}\pi$-lenses.

If necessary by considering a group conjugate to G we may suppose that $0 \in K_G(\alpha)$. It is easy to check that our results are invariant under conjugation. Construct the radius (i.e. geodesic) from 0 to the given limit point x. Suppose a point of this radius lay in $\Lambda_j(\frac{1}{2}\pi)$; as $0 \notin \Lambda_j(\frac{1}{2}\pi)$ it would follow that $x \in \Omega_j$. This is impossible and hence this radius lies entirely in $K_G(\frac{1}{2}\pi)$. In particular the point $x_1 = (1 - X^{-1})\, x \in K_G(\frac{1}{2}\pi)$.

Let $\{C_p\}$ be an admissible set of horocycles. Fix p. Then there is $C'_p \supset C_p$, a horocycle at p, so that

$$C'_p \supseteq (D \cap K_G(\tfrac{1}{2}\pi)) \smallsetminus \bigcup_q C_q \tag{5}$$

since the right hand side is relatively compact (see § 2). If q is a parabolic vertex we let

$$\begin{aligned} C'_q &= g(C_p) && \text{if} \quad q = g(p) \\ &= C_q && \text{if} \quad q \notin \{g(p) \mid g \in G\}. \end{aligned}$$

From (5) one finds that $\qquad \bigcup_q C'_q \supseteq K_G(\frac{1}{2}\pi)$.

In particular $x_1 \in \bigcup_q C'_q$. So for some q, $x_1 \in C'_q$. Thus there is $p_j \in P_0$ and $g \in G$ so that $x_1 \in g(C'_{pj})$.

The diameter of $g(C'_{p_j})$ is at least X^{-1} (by the construction of x_1) and hence

$$d(g(C'_{p_j})) \geqslant 1/(2X).$$

By lemma 4.7 (ii) we can find g_0 so that $g(C'_{p_j}) = g_0(C'_{p_j})$ and $\mu(g_0) \leqslant c_2 X$. Take $g = g_0$. By lemma 4.1 (i)

$$|x_1 - g(p_j)| \leqslant c_3/\sqrt{(X\mu(g))}.$$

As $|x_1 - x| = X^{-1}$ it then follows that

$$|x - g(p_j)| \leqslant (c_3 + \sqrt{c_2})/\sqrt{(X\mu(g))}.$$

The statement of the theorem follows on absorbing constants.

The proof of theorem 2 *is analogous.* Construct x_1 as above. Then $x_1 \in K_G(\frac{1}{2}\pi)$.

$D \cap K_G(\frac{1}{2}\pi)$ is relatively compact. Thus, for small enough $\beta > 0$,

$$C(\eta, \eta'; \beta) \supseteq D \cap K_G(\tfrac{1}{2}\pi).$$

Hence there is $g \in G$ so that

$$x_1 \in g(D \cap K_G(\tfrac{1}{2}\pi)) \subseteq gC(\eta, \eta'; \beta).$$

By lemma 4.2 (i),

$$|g(\eta) - g(\eta')| > c_4/X.$$

Hence, by lemma 4.8 (ii) we can find g_0 so $g_0 C(\eta, \eta'; \beta) = g\,C(\eta, \eta'; \beta)$ and $\mu(g_0) \leqslant c_5 X$. We can take $g = g_0$.

As $x_1 \in C(g\eta, g\eta'; \beta)$ it follows by lemma 4.2 (i) that

$$\min\,(|g\eta - x_1|, |g\eta' - x_1|) \leqslant c_6/X.$$

As $|x - x_1| = 1/X$ this implies that

$$\min\,(|g\eta - x|, |g\eta' - x|) \leqslant (c_6 + 1)/X.$$

The assertion of the theorem follows on absorbing constants.

8. Some estimates for groups of the first kind

In this section only groups of the first kind will be considered.

Our object is to convert the results of § 7 into quantitative estimates on the distribution of group elements. This is a necessary technical preparation for the next two sections.

We need the following two propositions.

Proposition 1. *Suppose G has parabolic elements and let $P_0 = \{p_1, \ldots, p_s\}$ be a complete set of parabolic vertices inequivalent under G. There are constants, c, c', c'', depending only on G and P_0 with the following property if $p, q \in P_0$, $g \in G \| G_q$ there is $h \in G \| G_p$ so that*

$$|h(p) - g(q)| < c/\mu(g)$$

and

$$c'\mu(g) \leqslant \mu(h) \leqslant c''\mu(g).$$

Proposition 2. *Suppose G has no parabolic elements and let η, η' be a conjugate pair of hyperbolic fixed points. There are constants c, c', c'', depending only on G and $\{\eta, \eta'\}$ with the property if*

$$g \in G \| G_{\eta\eta'} \quad \textit{there is} \quad h \in G \| G_{\eta\eta'}$$

so that

$$|g(\eta') - h(\eta)| < c/\mu(g)$$

and

$$c'\mu(g) \leqslant \mu(h) \leqslant c''\mu(g).$$

Proofs. Proposition 2 is merely a form of lemma 4.8 convenient for our present purposes; it is true if we take $h = g$.

Now we concentrate on proposition 1. For this we need a preliminary discussion. Let $C_1 = C(p_1, d_1)$, $C_2 = C(p_2, d_2)$ be horocycles as in § 4. As in § 5 we form the invariant

$$A(C_1, C_2) = |p_1 - p_2|^2 / d_1 d_2.$$

We observed that C_1, C_2 intersected if and only if $A(C_1, C_2) \leqslant 4$. This is easily refined; if $\partial C_1, \partial C_2$ meet at an angle ϕ then

$$A(C_1, C_2) = 4\cos^2 \tfrac{1}{2}\phi.$$

In particular, if $\partial C_1, \partial C_2$ meet at right angles

$$A(C_1, C_2) = 2.$$

Let $\zeta \in S$ and form

$$B(C_1, C_2, \zeta) = \frac{|\zeta - p_1|^2 d_2}{|\zeta - p_2|^2 d_1}.$$

If $\gamma \in \operatorname{con}(\Delta)$ $B(\gamma C_1, \gamma C_2, \gamma\zeta) = B(C_1, C_2, \zeta)$. Thus for any b, $\{\zeta\colon B(C_1, C_2, \zeta) \leqslant b\}$ defines invariantly an interval around p_1.

Let Q be the set of ζ in S so that the $\frac{1}{2}\pi$-line joining p_1 and ζ does not meet $(\partial C_1) \cap C_2$. This is also an invariantly defined interval around p_1. As in § 5 we can restrict our deliberations to the case $p_1 = 1, p_2 = -1, d_1 = d_2$. On examining this we find that

$$Q = \{\zeta | B(C_1, C_2, \zeta) \leqslant A(C_1, C_2)\}. \tag{1}$$

Now we can return to the proof of proposition 1. Let C_p, C_q be horocycles at p, q respectively so that ∂C_p, ∂C_q meet at right angles. Suppose π generates $G_{g(q)}$ and that π translates a point of $g(\partial C_q)$ a hyperbolic distance D along itself, D being independent of g.

We can find $k \in \mathbf{Z}$ so that the two intersections of $g(\partial C_q)$ and $\pi^k g(\partial C_p)$ are on the same side of the summit of $g(\partial C_q)$ and so that one of the points of intersection is within D of the summit.

Let us derive some consequences of these constructions. Firstly, as $g(\partial C_q)$ and $\pi^k g(\partial C_p)$ still meet at right angles

$$A(gC_q, \pi^k g C_p) = 2. \tag{2}$$

Let

$$g(C_q) = C(g(q), d_1),$$

$$\pi^k g(C_p) = C(\pi^k g(p), d_2).$$

By lemma 4.7 (ii) there is c_1, depending only on C_p, C_q and G so that

$$d_1, d_2 \leqslant c_1. \tag{3}$$

By lemma 4.7, as $g \in G \| G_q$, there are $c_2, c_3 > 0$ so that

$$c_2 \mu(g)^{-1} \leqslant d_1 \leqslant c_3 \mu(g)^{-1}. \tag{4}$$

Now, if w is a point of $\partial g(C_q)$ within a distance D of the summit

$$\frac{(1 - |1 - d_1|\,|w|/(1 + d_1))^2}{(1 - |w|^2)\,(1 - ((1 - d_1)/(1 + d_1))^2)} \leqslant \tfrac{1}{2}(1 + \cosh D)$$

from which one obtains

$$1 - |w|^2 \geqslant \frac{2^{-1}}{1 + \cosh D} \min(d_1, d_1^{-1}).$$

As there is such a point on $\partial\pi^k g(C_p)$ we obtain that

$$d_2 \geqslant \frac{8^{-1}}{1+\cosh D}\min(d_1, d_1^{-1}).$$

By (3) we see that there is $c_4 > 0$ so that

$$d_2 \geqslant c_4 d_1. \tag{5}$$

On the other hand, as both intersections $\partial(gC_q)$ and $\partial(\pi^k gC_p)$ are on the same side of the summit the $\frac{1}{2}\pi$-line through 0 does not pass through $\partial g(C_q) \cap \pi^k g(C_p)$. So by (1) and (2),

$$\frac{|g(q)+g(q)|^2}{|\pi^k g(p)+g(q)|^2}\frac{d^2}{d_1} \leqslant A(g(C_q), \pi^k g(C_p)) = 2.$$

As $|g(q)| = 1$ and $|\pi^k g(p)+g(q)| \leqslant 2$ we find

$$d_2 \leqslant 2d_1. \tag{6}$$

Define h by $h \in G\|G_p$ and $hG_p = \pi^k gG_p$. By lemma 4.7 there are constants $c_5, c_6 > 0$ so that

$$c_5\mu(h)^{-1} \leqslant d_2 \leqslant c_6\mu(h)^{-1}. \tag{7}$$

So, by (4), (5), (6), (7) $\quad c_5 2^{-1}c_3^{-1}\mu(g) \leqslant \mu(h) \leqslant c_6 c_4^{-1}c_2^{-1}\mu(g).$ (8)

On the other hand, $hC_p = \pi^k g(C_p)$ and so as $A(hC_p, gC_q) = 2$,

$$\begin{aligned}|h(p)-g(q)|^2 &= 2d_1 d_2 \\ &\leqslant 4d_1^2 \\ &\leqslant 4c_3^2\mu(g)^{-2}.\end{aligned} \tag{9}$$

Equations (8) and (9) constitute the assertion of proposition 1 which is thereby proved.

It is now necessary to introduce some notation. For $a \in S$ we set

$$B(a,r) = \{y \mid y \in S, |a-y| < r\}.$$

Let $I(a,r)(x)$ be the characteristic function of $B(a,r)$ on S. Let us call a set of the form $B(a,r)$ or $\bar{B}(a,r)$ an *interval*. If J is an interval let $|J|$ be the angular measure of J. As long as $r \leqslant 2$

$$|B(a,r)| = 4\arcsin(\tfrac{1}{2}r).$$

As $2\phi/\pi \leqslant \sin\phi \leqslant \phi$ $(0 \leqslant \phi \leqslant \frac{1}{2}\pi)$ we find, if $r \leqslant 2$,

$$2r \leqslant |B(a,r)| \leqslant \pi r. \tag{10}$$

Now we can state the major results of this section.

THEOREM 1. *Assume G has parabolic elements. Let p be a parabolic vertex. There are constants $k, c_1, c_2, c_3, c_4 > 0$ depending only on G and p so that $k \in]0, 1[$ and, if J is an interval then*

$$\text{(i)} \quad \sum_{g\in B} 1 \leqslant c_1|J|X + c_2, \tag{11}$$

where $B = \{g \in G \mid g(p) \in J, kX < \mu(g) \leqslant X\}$,

$$\text{(ii)} \quad \sum_{g\in C} 1 \geqslant c_3|J|X - c_4\sqrt{X}, \tag{12}$$

where $C = \{g \in G\|G_p \mid g(p) \in J, kX < \mu(g) \leqslant X\}$.

THEOREM 2. *Assume G has no parabolic elements. Let η be a hyperbolic fixed point. There are constants k, c_1, c_2, c_3, c_4 depending only on G and η so that $k \in]0, 1]$ and, if J is an interval then*

$$\text{(i)} \quad \sum_{g\in B} 1 \leqslant c_1 |J|\, X + c_2, \tag{13}$$

where $B = \{g \in G \,|\, g(\eta) \in J,\ kX < \mu(g) \leqslant X\}$,

$$\text{(ii)} \quad \sum_{g\in C} 1 \geqslant c_3 |J|\, X - c_4, \tag{14}$$

where $C = \{g \in G \| G_\eta \,|\, g(\eta) \in J,\ kX < \mu(g) \leqslant X\}$.

These two theorems contain almost everything of their kind which has been obtained by purely geometrical means. In some respects they can be improved by a more advanced analytic theory but a discussion of what can be obtained is out of place here.

That we have used a fixed k is merely a matter of convenience; the value of k is not relevant.

Theorem 1 is the hardest of these two theorems and we will only give the proof for it. The other requires no new ideas.

Proof of theorem 1. Let P_0 be a complete set of inequivalent parabolic vertices containing p. By theorem 7.1 there are constants c_5, c_6 so that, on S,

$$1 \leqslant \sum_{q\in P_0} \sum_{g\in G\|G_q} I(g(q), c_5/\surd(\mu(g)\, X)) \leqslant c_6, \tag{15}$$

Suppose $J = B(a,r)$ and let $\qquad J_\pm = B(a, r \pm 2c_5 X^{-\frac{1}{2}})$,

where we set $J_- = \varnothing$ if $r \leqslant 2c_5 X^{-\frac{1}{2}}$. The distinction between open and closed intervals will not concern us here. Let

$$B'_q = \{g \in G \| G_q \,|\, g(q) \in J,\ \mu(g) \leqslant X\}.$$

Then, with a little thought, (15) yields

$$\sum_{q\in P_0} \sum_{g\in B'_q} I(g(q), c_5/\surd(\mu(g)\, X)) \begin{cases} \leqslant c_6 & (\text{on } S) \\ = 0 & (\text{on } S \smallsetminus J_+) \\ \geqslant 1 & (\text{on } J_-). \end{cases}$$

When we integrate over S and use (10) we see that for some constants c_7, c_8, $c_9 > 0$

$$\sum_{q\in P_0} \sum_{g\in B'_q} (\mu(g)\, X)^{-\frac{1}{2}} \geqslant c_7 |J| - c_9 X^{\frac{1}{2}}, \tag{16a}$$

$$\leqslant c_8 |J| + c_9 X^{\frac{1}{2}}. \tag{16b}$$

(16*b*) is not precise enough and we must obtain a better variant. Let k be so that $0 < k < 1$. Let

$$J_1 = B\left(a, r + \frac{c_5}{\sqrt{k}} X^{-1}\right)$$

and $\qquad B''_q = \{g \in G \| G_q \,|\, g(q) \in J,\ kX < \mu(g) \leqslant X\}.$

From (15) one obtains

$$\sum_{q\in P_0} \sum_{g\in B''_q} I(g(p), c_5/\surd(\mu(g)\, X)) \begin{cases} \leqslant c_6 & (\text{on } S) \\ = 0 & (\text{on } S \smallsetminus J_1). \end{cases}$$

If we integrate this over S and use (10) we obtain

$$\sum_{q\in P_0} \sum_{g\in B''_q} (\mu(g)\, X)^{-\frac{1}{2}} \leqslant c_{10} |J| + c_{11} X^{-1}. \tag{17}$$

The constants depend on G, P_0 and c_{11} depends also on k.

To proceed we need the following lemma which we shall prove at the end of the section.

LEMMA. *Let q be a parabolic vertex. There is a constant ν_q depending only on q with the following property: if $g \in G\|G_q$, $Y \geqslant \mu(g)$ the set $\{\gamma \in G_q | \mu(g\gamma) \leqslant Y\}$ has at most $\nu_q(Y/\mu(g))^{\frac{1}{2}}$ elements.*

Now let $c_{12} = \max_{q \in P_0}(\nu_q)$. Let $n(g, q; Y)$ be the number of elements of

$$\{\gamma \in G_q | \mu(g\gamma) \leqslant Y\}.$$

Then by the lemma and (17) we obtain

$$\sum_{q \in P_0} \sum_{g \in B''_q} \frac{1}{X} n(g, q; X) \leqslant c_{12} c_{10} |J| + c_{12} c_{11} X^{-1}.$$

Thus if we let $$B^*_q = \{g \in G | g(q) \in J, kX < \mu(g) \leqslant X\}$$
we have $$\sum_{q \in P_0} \sum_{g \in B^*_q} 1 \leqslant c_{12} c_{10} |J| X + c_{12} c_{11}.$$

The summands are all positive and so, for $q \in P_0$,

$$\sum_{g \in B^*_q} 1 \leqslant c_{12} c_{10} |J| X + c_{12} c_{11}.$$

In particular this holds for p; this gives (11) and proves part (i) of theorem 1.

Now let $$B'_q(t) = \{g \in G\|G_q | g(q) \in J, \mu(g) \leqslant tX\}.$$
By (16 *b*) $$\sum_{q \in P_0} \sum_{g \in B'_q(t)} \mu(g)^{-\frac{1}{2}} \leqslant c_8 |J| t^{\frac{1}{2}} X^{\frac{1}{2}} + c_9. \tag{18}$$

Now set $t = (c_7/2c_8)^2$ and subtract (18) from (16 *a*). If we set

$$B''_q(t) = \{g \in G\|G_q | g(q) \in J, tX < \mu(g) \leqslant X\}$$
we find $$\sum_{q \in P_0} \sum_{g \in B''_q(t)} \mu(g)^{-\frac{1}{2}} \geqslant \tfrac{1}{2} c_7 |J| X^{\frac{1}{2}} - 2c_9.$$

From this it follows at once that

$$\sum_{q \in P_0} \sum_{g \in B''_q(t)} 1 \geqslant \tfrac{1}{2} c_7 t^{\frac{1}{2}} |J| X - 2c_9 t^{\frac{1}{2}} X^{\frac{1}{2}}. \tag{19}$$

We are now in a position to apply proposition 1. According to this there are constants c_{13}, c_{14}, c_{15} so that, if $q \in P_0$, $g \in G\|G_q$ then there is $h \in G\|G_p$ so that

$$|h(p) - g(q)| \leqslant c_{13} \mu(g)^{-1} \tag{20}$$
and $$c_{14} \mu(g) \leqslant \mu(h) \leqslant c_{15} \mu(g). \tag{21}$$

Let us make a further observation. Fix $t \in]0, 1[$. Then by theorem 7.1 if $q_1, q_2 \in P_0$, $g_1 \in G\|G_{q_1}$, $g_2 \in G\|G_{q_2}$ and
$$tX < \mu(g_1), \mu(g_2) \leqslant X$$

there is $c_{16} > 0$ so that, if $g_1(q_1) \neq g_2(q_2)$,

$$|g_1(q_1) - g_2(q_2)| > c_{16}/X. \tag{22}$$

Let $J^- = B(a, r - c_{13} X^{-1})$

$$B^-_q = \{q \in G\|G_q | g(q) \in J^-, kX < \mu(g) \leqslant X\}$$
and $$B^\dagger = \{h \in G\|G_p | h(p) \in J, c_{14} kX < \mu(g) \leqslant c_{15} X\}.$$

Then we shall show that $$\operatorname{card}(B^\dagger) \geqslant \left(\frac{kc_{13}}{c_{16}} + 1\right)^{-1} \operatorname{card}(B^-_q). \tag{23}$$

For if $g \in B_q^-$ there is, by proposition 1, h satisfying (20) and (21). By (20), for such a $h, h \in B^\dagger$. On the other hand, by (22) not more that $c_{13}k/c_{16}+1$ elements of B_q^- can give rise in this way to the same element of $B^\dagger$; (23) follows at once.

Suppose P_0 has s elements. Choose $k = t$. Then (19) and (23) yield

$$s\,\mathrm{card}\,(B^\dagger) \geqslant \left(\frac{kc_{13}}{c_{16}}+1\right)^{-1} \tfrac{1}{2}c_7 t^{\frac{1}{2}}\,|J^-|\,X - 2c_9 t^{\frac{1}{2}} X^{\frac{1}{2}}.$$

As, by (10) $$|J^-| \geqslant |J| - 2\pi c_{13} X^{-1}$$
we have that, for suitable c_{17}, c_{18}

$$\mathrm{card}\,(B^\dagger) \geqslant c_{17}\,|J|\,X - c_{18} X^{\frac{1}{2}}.$$

This is the assertion of theorem 1 (ii). So theorem 1 is completely proved except for the proof of the lemma.

Proof of lemma. Let A be the bilinear map, mapping Δ on to $\boldsymbol{H}$, p to ∞ and 0 to i. Then we define, for $u \in \mathrm{con}\,(\boldsymbol{H})$, u represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $a, b, c, d \in \boldsymbol{R}$, $ad-bc = 1$,

$$\mu_*(u) = a^2+b^2+c^2+d^2.$$

One checks that if $v \in \mathrm{con}\,(\Delta)$ that $\quad \mu(v) = \mu_*(AvA^{-1})$.

Let $G^A = AGA^{-1}$ and suppose that G^A_∞ is generated by $z \mapsto z+\lambda$. Let $g \in G^A \| G^A_\infty$ and suppose g is represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that $ad-bc = 1$. Then

$$\mu_*\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & n\lambda \\ 0 & 1 \end{pmatrix}\right) = (a^2+b^2+c^2+d^2) + 2(ab+cd)\,\lambda n + (a^2+c^2)\,\lambda^2 n^2.$$

As $g \in G^A \| G^A_\infty$ this takes its minimal value at $n = 0$; hence

$$\frac{|ab+cd|}{(a^2+c^2)} \leqslant \tfrac{1}{2}\lambda.$$

Since $ad-bc = 1$ this gives $$\left|\frac{d}{c} - \frac{a}{c(a^2+c^2)}\right| \leqslant \tfrac{1}{2}\lambda.$$

There is $c_{19} > 0$ so that if $c \neq 0$, $|c| > c_{19}$ (Lehner 1964, p. 88). If $c = 0$ the problem is easy as $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus

$$\begin{aligned}\left|\frac{d}{c}\right| &\leqslant \tfrac{1}{2}\lambda + \frac{1}{|c|}\frac{|a|}{a^2+c^2} \\ &\leqslant \tfrac{1}{2}\lambda + |c|^{-2} \\ &\leqslant \tfrac{1}{2}\lambda + c_{19}^{-2}.\end{aligned}$$

Likewise, $$\left|\frac{b}{a} + \frac{c}{a(a^2+c^2)}\right| \leqslant \tfrac{1}{2}\lambda.$$

Thus, if $c \neq 0$, $$\begin{aligned}|b| &\leqslant \tfrac{1}{2}\lambda\,|a| + \frac{|c|}{a^2+c^2} \\ &\leqslant \tfrac{1}{2}\lambda\,|a| + |c|^{-1} \\ &\leqslant \tfrac{1}{2}\lambda\,|a| + c_{19}^{-1} \\ &\leqslant (\tfrac{1}{2}\lambda + c_{19}^{-2})\,(|a|+|c|).\end{aligned}$$

Hence $$b^2+d^2 \leqslant (\tfrac{1}{2}\lambda+c_{19}^{-2})^2\,((|a|+|c|)^2+|c|^2)$$
$$\leqslant (\tfrac{1}{2}\lambda+c_{19}^{-2})^2\,3(a^2+c^2).$$

Thus $$(a^2+c^2) \geqslant (1+3(\tfrac{1}{2}\lambda+c_{19}^{-2})^{-2})^{-1}\,\mu_*(g). \tag{24}$$

Set $$c_{20} = (1+3(\tfrac{1}{2}\lambda+c_{19}^{-2})^{-2})^{-1}.$$

Define A, B, C by writing $$\mu_*\left(g\begin{pmatrix}1 & n\lambda\\ 0 & 1\end{pmatrix}\right) = A+2Bn+Cn^2$$

so $$A = \mu_*(g),\quad B = (ab+cd)\,\lambda,\quad C = (a^2+c^2)\,\lambda^2.$$

The number we are interested in is the number of solutions of

$$Cn^2+2Bn+A \leqslant Y.$$

From its definition the left hand side, as a function of n, has no real roots and a minimum in $[-\tfrac{1}{2}, +\tfrac{1}{2}]$. Thus it can be written as

$$C(n+\theta')^2+A' \leqslant Y,$$

where $|\theta'| \leqslant \tfrac{1}{2}, A' > 0$. This has at most

$$2\left(\frac{Y-A'}{C}\right)^{\frac{1}{2}}+1$$

solutions. Thus by (24) this is at most

$$2c_{20}^{-\frac{1}{2}}\,\mu(g)^{-\frac{1}{2}}\,Y^{\frac{1}{2}}+1.$$

As $Y \geqslant \mu(g)$ this is at most $$(2c_{20}^{-\frac{1}{2}}+1)\,(Y/\mu(g))^{\frac{1}{2}},$$
which proves the lemma.

9. Metric theorems

In this section we continue the investigations begun in § 3 but now from the view-point of measure theory. Our model is the so called 'metric theory' of diophantine approximation (see Cassels 1965, ch. VII). Our conclusions will not be quite as sharp as the theorems on which they are modelled but for 'practical' purposes they are just as good.

Throughout this section G is of the first kind.

Theorem. *Let w be a positive decreasing function on $[2,\infty[$ and suppose there is $c > 0$ so that*

$$w(2x)/w(x) > c. \tag{1}$$

Let y be a parabolic vertex, if there are any, or a hyperbolic fixed point if there are none. Let $A(y)$ be the set of $x\in S$ for which
$$|x-g(y)| \leqslant w(\mu(g))/\mu(g) \tag{2}$$
can be solved infinitely often for $g\in G$. Then

(i) *if for some $K > 1$,* $\sum_{n=1}^{\infty} w(K^n) < \infty$ *$A(y)$ is of measure* 0

(ii) *if for some $K > 1$,* $\sum_{n=1}^{\infty} w(K^n) = \infty$ *$S\smallsetminus A(y)$ is of measure* 0.

Before proving this let us remark that by Cauchy's Condensation test if w is decreasing then

$$\sum_{n=2}^{\infty} w(n^2)/n$$

and, if $K > 1$, $$\sum_{n=1}^{\infty} w(K^n)$$

converge or diverge together. The first of these is the classical series. It also follows that in the assumptions of (i) and (ii) the value of K is immaterial.

Let us also note that from (1), if $T > 1$, there is a constant $c(T) > 0$ so that

$$w(Tx)/w(x) > c(T). \tag{3}$$

It is a consequence of the theorem that if

$$w(x) = (\ln x)^{-1}$$

then the assumption of (ii) holds. Thus for almost all points x of S

$$|x-g(y)| < 1/(\mu(g)\ln\mu(g))$$

can be solved infinitely often with $g \in G$.

Except for (1) our conditions on w are the same as those in metric number theory and (1) is not a very great restriction. Thus this theorem is not much weaker than the classical one. Also the example with $w(x) = (\ln x)^{-1}$ shows that our theorem answers the problem posed by Lehner at the end of chapter X of Lehner (1964).

Proof. We prove (i). Regard S as a probability space with the Lebesgue measure normalized to have total mass 1; denote this by P. Let

$$A_n = \{x \in S| \quad \text{there is} \quad g \in G\colon K^n < \mu(g) \leqslant K^{n+1} \quad \text{and} \quad |x-g(y)| < w(\mu(g))/\mu(g)\}.$$

Then, if we set
$$A(g) = \{x \in S|\, |x-g(y)| < w(\mu(g))/\mu(g)\}.$$

A_n is the union of $A(g)$ with $K^n < \mu(g) \leqslant K^{n+1}$. Clearly

$$P(A(g)) \leqslant w(K^n)/K^n.$$

By theorem 8.1 or 8.2 there are at most c_1K^{n+1} such intervals. Hence

$$P(A_n) \leqslant c_1 K w(K^n).$$

So, by the assumption of (i)
$$\sum_{n=1}^{\infty} P(A_n) < \infty.$$

So, by the first Borel–Cantelli lemma the set of points in an infinity of A_n is of measure 0. This proves (i).

This leaves (ii) which is much less trivial. Let us observe that we need only prove it when $w(x) \to 0$ as $x \to \infty$ since the class of such functions is non-empty.

We have to make use of two standard propositions.

Proposition 1. *Let T be a measurable subset of S and suppose that for any $g \in G$, $g(T) = T$ (at least up to a set of measure 0). Then*
$$P(T) = 0 \quad \textit{or} \quad 1.$$
For a proof see Lehner (1964, p. 322).

Proposition 2. *Let A be a countable set and let $\{J_\alpha|\alpha \in A\}$ and $\{J'_\alpha|\alpha \in A\}$ be two sets of intervals whose radii tend to 0. Suppose that the centres of J_α and J'_α are the same, and that $|J_\alpha|/|J'_\alpha|$ is constant. If J_∞ (resp. J'_∞) is the set of points belonging to infinitely many J_α (resp. J'_α) then*

$$P(J_\infty) = P(J'_\infty).$$

For a proof see Cassels (1950, lemma 9).

From these we will deduce the following proposition:

Proposition 3. *Let w be a positive decreasing function on $[2, \infty[$ satisfying (1).*

Let $y \in S$ and let $A(y, w)$ be the set of $x \in S$ for which

$$|x - g(y)| < w(\mu(g))/\mu(g)$$

can be solved infinitely often with $g \in G$. Then

$$P(A(y, w)) = 0 \quad or \quad 1.$$

Proof. Let k be positive and let $A_k = A(y, kw)$.

By proposition 2 $$P(A_k) = P(A_1). \tag{4}$$

Let $\gamma \in G$. Then $$\gamma(A_k) = \{\gamma(x) \,|\, |g(y) - x| < kw(\mu(g))/\mu(g) \text{ inf. often}\}$$
$$= \{x \,|\, |g(y) - \gamma^{-1}(x)| < kw(\mu(g))/\mu(g) \text{ inf. often}\}.$$

But as γ is a diffeomorphism from S to S there is a number $d_1(\gamma) > 0$ so that

$$|\gamma g(y) - x| \geqslant d_1(\gamma)\, |g(y) - \gamma^{-1}(x)|.$$

So $$\gamma(A_k) \supseteq \{x \,|\, |\gamma g(y) - x| \leqslant d_1(\gamma)\, kw(\mu(g))/\mu(g) \text{ inf. often}\}.$$

We know that $$\mu(\gamma g) \geqslant \mu(\gamma)^{-1} \mu(g).$$

From this and (1) there is $d_2(\gamma) > 0$ so that

$$w(\mu(\gamma g)) \geqslant d_2(\gamma)\, w(\mu(g)).$$

Thus $$\gamma(A_k) \supseteq A_{d_2(\gamma)\, k/\mu(\gamma)}. \tag{5}$$

Expressions (4) and (5) imply that, up to a set of measure 0,

$$\gamma(A_1) = A_1.$$

So by proposition 1 $$P(A_1) = 0 \quad \text{or} \quad 1$$
which completes the proof.

Now let k be a real number in $]0, 1[$ so that the second part of theorem 8.1 or 8.2 is true for the y of the statement of the theorem of this chapter. Let us remark that we may assume, for any given $c_2 > 0$ that $w(x) \leqslant c_2$. Now set, for $K = k^{-1}$,

$$A_n = \{x \in S| \quad \text{there is} \quad g \in G\colon K^n < \mu(g) \leqslant K^{n+1} \quad \text{and} \quad |x - g(y)| < w(\mu(g))/\mu(g)\}$$

and $$A(g) = \{x \in S|\, |x - g(y)| < w(\mu(g))/\mu(g)\}.$$

A_n is the union of $A(g)$ with $K^n < \mu(g) \leqslant K^{n+1}$. By (1) and the fact that w is decreasing there are constants $c_3, c_4 > 0$ so that

$$c_3\, w(K^n)\, K^{-n} \leqslant P(A(g)) \leqslant c_4\, w(K^n)\, K^{-n}. \tag{6}$$

From theorem 8.1 or 8.2 it follows that there is a constant $c_5 > 0$ so that if $w(x) \leqslant c_5$, and if $K^n < \mu(g) \leqslant K^{n+1}$, $K^n < \mu(h) \leqslant K^{n+1}$ then $A(g)$ and $A(h)$ only meet if $g(y) = h(y)$. By theorem 8.1 or 8.2 it follows that there are constants c_6, $c_7 > 0$ so that

$$c_6 K^n \leqslant \text{card}\{g(y) \,|\, K^n < \mu(g) \leqslant K^{n+1}\} \leqslant c_7 K^n. \tag{7}$$

We shall assume henceforth that $w(x) \leqslant c_5$. Expressions (6) and (7) imply that

$$c_3 c_6\, w(K^n) \leqslant P(A_n) \leqslant c_4 c_7\, w(K^n). \tag{8}$$

Consider now $A_n \cap A_{n+m} (m > 0)$. By theorem 8.1 or 8.2 and (6) each $A(g)$ with $K^n < \mu(g) \leqslant K^{n+1}$ meets at most, for some constant $c_8 > 0$,

$$c_8(w(K^n)\, K^{-n} K^{n+m} + 1)$$

intervals $A(h)$ with $K^{n+m} < \mu(h) \leqslant K^{n+m+1}$. Thus, by (6) the intersection of A_{n+m} with

$$A(g) \qquad (K^n < \mu(g) \leqslant K^{n+1})$$

has probability at most $c_4 c_8 (w(K^n)\, K^{-n} K^{n+m} + 1)\, w(K^{n+m})\, K^{-(n+m)}$.

By (7) A_n is made up of at most $c_6 K^n$ intervals $A(g)$. So

$$\begin{aligned} P(A_n \cap A_{n+m}) &\leqslant c_4 c_6 c_8 (w(K^n)\, K^m + 1)\, w(K^{n+m})\, K^{-m} \\ &= c_4 c_6 c_8 (w(K^n)\, w(K^{n+m}) + w(K^{n+m})\, K^{-m}). \end{aligned}$$

Thus
$$\begin{aligned} \sum_{\substack{n \geqslant 0 \\ m > 0 \\ n+m \leqslant N}} P(A_n \cap A_{n+m}) &\leqslant c_4 c_6 c_8 \Big(\sum_{\substack{n \geqslant 0 \\ m > 0 \\ n+m \leqslant N}} (w(K^n)\, w(K^{n+m}) + w(K^{n+m})\, K^{-m}) \Big) \\ &\leqslant c_4 c_6 c_8 \Big(\sum_{\substack{n \geqslant 0 \\ m > 0 \\ n+m \leqslant N}} w(K^n)\, w(K^{n+m}) + \sum_{n \leqslant N} w(K^n)\, (K-1)^{-1} \Big). \end{aligned}$$

As
$$\sum_{n,\, m \leqslant N} P(A_n \cap A_m) = \sum_{n \leqslant N} P(A_n) + 2 \sum_{i < j \leqslant N} P(A_i \cap A_j)$$

we find
$$\sum_{n,\, m \leqslant N} P(A_n \cap A_m) \leqslant c_4 c_6 c_8 \Big(\sum_{n \leqslant N} w(K^n) \Big)^2 + (1 + c_4 c_6 c_8 (K-1)^{-1}) \sum_{n \leqslant N} w(K^n).$$

Now let
$$M_N = \sum_{n \leqslant N} P(A_n).$$

By (8) and the inequality above there are constants c_9, c_{10} so that

$$\sum_{n,\, m \leqslant N} P(A_n \cap A_m) \leqslant c_9 M_N^2 + c_{10} M_N. \tag{9}$$

Also, by the assumption of (ii) and (8), as $N \to \infty$

$$M_N \to \infty. \tag{10}$$

We now need an extension of the second Borel–Cantelli lemma.

Proposition 4. *Let (Ω, P) be a probability space. Let A_n be a sequence of events and c a positive number so that*

$$\sum_{n,\, m \leqslant N} P(A_n \cap A_m) \leqslant c \Big(\sum_{n \leqslant N} P(A_n) \Big)^2 \tag{11}$$

and

$$\sum_{n=1}^{\infty} P(A_n) = \infty. \tag{12}$$

Let A_∞ be the set of $x \in \Omega$ so that x lies in infinitely many A_n. Then

$$P(A_\infty) \geqslant c^{-1}.$$

Let us assume this for the moment. We shall apply it to the space (S, P) and the sequence of events A_n considered above. (10) shows that (12) is satisfied. Also (9) and (10) show that there is some constant c so that (11) is satisfied. Thus $P(A_\infty) > 0$. But, by definition $A(y) \supseteq A_\infty$. So

$$P(A(y)) > 0.$$

Hence by proposition 3 $P(A(y)) = 1$. This completes the proof of the theorem.

The proof of proposition 4 is based on familiar ideas which can be found in Cassels (1965, ch. VII) and Chung (1960). The proof depends on the following lemma (due in the first place to Paley & Zygmund).

LEMMA. *Let X be a non-negative random variable; let $\mu = EX$, $r^2 = E(X^2)$. Then if $b \leqslant \mu/r$*

$$P\{X \geqslant br\} \geqslant ((\mu/r)-b)^2.$$

Proof. This is the same as in Cassels (1965, p. 112).

Let I_A be the indicator (characteristic) function of A. Set

$$f_n = \sum_{j \leqslant n} I_{A_j}.$$

Then
$$Ef_n = \sum_{j \leqslant n} P(A_j).$$

Also
$$f_n^2 = \sum_{i,j \leqslant n} I_{A_i \cap A_j}.$$

So
$$E(f_n^2) = \sum_{i,j \leqslant n} P(A_i \cap A_j)$$
$$\leqslant c(Ef_n)^2.$$

Let us apply the lemma to f_n. Choose $b < c^{-\frac{1}{2}}$. Then

$$P\{f_n \geqslant bE(f_n^2)^{\frac{1}{2}}\} \geqslant (c^{-\frac{1}{2}}-b)^2.$$

Let
$$S_n = \{f_n \geqslant bE(f_n^2)^{\frac{1}{2}}\}.$$

By the Cauchy–Schwartz inequality
$$E(f_n^2) \geqslant (Ef_n)^2.$$

Thus, if $x \in \Omega$ is in infinitely many S_n then

$$f_n \geqslant bE(f_n)$$

infinitely often. Hence, as $Ef_n \rightarrow \infty$ (by (12)) x belongs to an infinity of A_n.

The set of points in an infinity of S_n is

$$\bigcap_N \bigcup_{n \geqslant N} S_n.$$

But
$$P(\bigcup_{n \geqslant N} S_n) \geqslant (c^{-\frac{1}{2}}-b)^2.$$

As Ω is a probability space
$$P(\bigcap_N \bigcup_{n \geqslant N} S_n) \geqslant (c^{-\frac{1}{2}}-b)^2.$$

Thus, as b is arbitrary it follows that
$$P(A_\infty) \geqslant c^{-1}.$$

This completes the proof.

10. Badly approximable limit points

In this section we complete our description of the rates of approximation which can be achieved in a Fuchsian group of the first kind. Our object is to show that the general approximation theorem 3.2 cannot be sharpened. That this is so in the narrowest sense follows from the results of § 6. In § 9 we showed that the set of points having the worst possible rate of approximation had measure 0; there we show that it is as large as possible under this restriction.

A point $x \in L_G$ is said to be *badly approximable* with respect to a finite set $A \subseteq L_G$ if there is $c = c(x) > 0$ so that for all $a \in A$, $g \in G$

$$|g(a)-x| > c/\mu(g).$$

As a metric on S we shall use $\rho(a,b) = |\arg(a\bar{b})|$

where arg lies in $]-\pi, +\pi]$. Clearly

$$(2/\pi)\,|a-b| \leqslant \rho(a,b) \leqslant |a-b|.$$

In this section G is of the first kind.

THEOREM. *Let A be a finite set of parabolic vertices, if there are any, and a finite set of hyperbolic fixed points otherwise. Then the set of points badly approximable with respect to A has Hausdorff dimension* 1.

This is closely related to a famous theorem of Jarník (1928). If G is of the second kind a similar construction to the one used below can be used to show that the corresponding set is uncountable. This shall not be proved here.

Proof. We shall prove the theorem only in the case that G has parabolic elements. The other case is similar and the construction is by no means as delicate.

It is clear that we may suppose the elements of A to be inequivalent. Hence A may be assumed to be a complete set of inequivalent parabolic vertices; $A = \{p_1, \ldots, p_s\}$, say. If p is any parabolic vertex then there is $p_i \in A$ and $g \in G \| G_{p_i}$ so that $p = g(p_i)$. The value $\mu(g)$ is determined uniquely and we write $\mu(p)$ for it. Let V be the set of all parabolic vertices. Then theorem 7.1 can be restated as:

Given $x \in L_G = S$ and $X > 2$ there is $p \in V$ with

$$\mu(p) < X,$$

$$\rho(x,p) \leqslant c/(\mu(p)\,X)^{\frac{1}{2}}. \tag{1}$$

If $p, q \in V$, $p \neq q$ then $$\rho(p,q) \geqslant c'/(\mu(p)\,\mu(q))^{\frac{1}{2}}. \tag{2}$$

We are now going to construct a subset of the set of badly approximable limit points as a Cantor set whose Hausdorff dimension can be made as close as we please to 1. The burden of the proof falls on a construction which is somewhat intricate. It will depend on two parameters ϵ, K whose possible ranges of values will be restricted as the argument progresses. ϵ will be sufficiently small and K sufficiently large.

Let $$V_n = \{p \in V | K^{n-1} \leqslant \mu(p) < K^n\}$$

and $$W_n = \bigcup_{m \leqslant n} V_m = \{p \in V | \mu(p) < K^n\}.$$

We define inductively a set of intervals $\{S(i_1, \ldots, i_n)\}$, where each i_j runs through a uniformly bounded set. Also, when $S(i_1, \ldots, i_n, j)$ is defined

$$S(i_1, \ldots, i_n, j) \subseteq S(i_1, \ldots, i_n).$$

We define also a finite subset $R(i_1, \ldots, i_n)$ of $S(_1, \ldots, i_n)$. This has at least three points. The $S(i_1, \ldots, i_n, j)$ are contained in the components of

$$S(i_1, \ldots, i_n) \smallsetminus R(i_1, \ldots, i_n).$$

As a formal start take $S(\varnothing) = S$. We will describe in detail the inductive step although it will take some time to verify its validity.

Suppose $S(i_1, \ldots, i_n)$ is defined. Set

$$R(i_1, \ldots, i_n) = S(i_1, \ldots, i_n) \cap V_{n+1}. \tag{3}$$

This is finite as V_{n+1} is finite. We shall see that if K is sufficiently large $R(i_1, \ldots, i_n)$ has at least three elements.

There are two cases to be considered. Suppose first that $n = 0$. Then $S(\varnothing) = S$ and $R(\varnothing) = V_1$. $S(\varnothing) \setminus R(\varnothing)$ is a disjoint union of a finite number of intervals and by the claim made above there are at least three. Let J be one such interval and let q_0, q_1 be the end-points of J. Then consider the interval

$$J_\epsilon = \{x \in J \mid \min_{j=0,1} (\rho(x, q_j)) \geqslant \tfrac{1}{2}\epsilon\rho(q_0, q_1)\}.$$

This is non-empty if $\epsilon < 1$. We will suppose that $\epsilon < 1$ henceforth. The $\{S(i_1)\}$ we define to be the $\{J_\epsilon\}$ as J runs through the components of $S(\varnothing) \setminus R(\varnothing)$. Thus $\{S(i_1)\}$ is a set of intervals at least three in number.

Now we can deal with the inductive step – that is, $n > 0$. As $S(i_1) \neq S$ it follows that

$$S(i_1, \ldots, i_n) \neq S.$$

Thus we can define a branch of arg on $S(i_1, \ldots, i_n)$. The set $R(i_1, \ldots, i_n)$ can be ordered by writing $r_1 < r_2$ if $\arg(r_1) < \arg(r_2)$. As $S(i_1, \ldots, i_n)$ is an interval $R(i_1, \ldots, i_n)$ is totally ordered by this order and we can talk about 'neighbouring' points of $R(i_1, \ldots, i_n)$. If $p, q \in R(i_1, \ldots, i_n)$ let $\overline{pq}$ be the open interval lying between them in $S(i_1, \ldots, i_n)$. The intervals $S(i_1, \ldots, i_n, j)$ are the intervals, formed from neighouring $p, q \in R(i_1, \ldots, i_n)$ as

$$\{x \in \overline{pq} \mid \min(\rho(x,p), \rho(x,q)) > \tfrac{1}{2}\epsilon\rho(p,q)\}.$$

If $\epsilon < 1$ all such intervals are non-empty and disjoint. As $R(i_1, \ldots, i_n)$ has at least three points there are at least two intervals $S(i_1, \ldots, i_n, j)$. The $S(i_1, \ldots, i_n, j)$ can be ordered in any manner – we need only that j runs through a finite set.

This gives the definition of the $S(i_1, \ldots, i_n)$. However, we still have to justify the steps and obtain some properties of the $S(i_1, \ldots, i_n)$.

Let us say that $x \in S$ is under the influence of p at level n if, for c as in (1),

$$\rho(x,p) \leqslant c(\mu(p)\, K^n)^{-\frac{1}{2}}.$$

We know that, by (1), every $x \in S$ is under the influence at level n of some p with $\mu(p) < K^n$, i.e. $p \in W_n$. Observe that if x is not under the influence of p at level n then it is not under the influence of p at level m if $m \geqslant n$. We show that, if K is sufficiently large, that the points of $S(i_1, \ldots, i_n)$ are not under the influence of any $v \in W_n$ at level $n+1$. For let $x \in S(i_1, \ldots, i_n)$ and suppose that x is under the influence of $v \in W_n$ at level $n+1$. From the definition of $S(i_1, \ldots, i_n)$ there are

$$p,\ q \in R(i_1, \ldots, i_{n-1})$$

which are neighbouring and so that

$$S(i_1, \ldots, i_n) = \{x \in \overline{pq} \mid \min(\rho(x,p), \rho(x,q)) > \tfrac{1}{2}\epsilon\rho(p,q)\}.$$

As
$$R(i_1, \ldots, i_{n-1}) = S(i_1, \ldots, i_{n-1}) \cap V_n$$

it follows that no point of V_n lies in $S(i_1, \ldots, i_n)$. So, by induction, no point of W_n lies in $S(i_1, \ldots, i_n)$. Thus

$$R(i_1, \ldots, i_{n-1}) = S(i_1, \ldots, i_{n-1}) \cap W_n.$$

Thus $v \notin \overline{pq}$, and as $x \in \overline{pq}$,
$$\rho(v,x) \geqslant \min(\rho(x,p), \rho(x,q))$$

and
$$\rho(v,x) \geqslant \min(\rho(v,p), \rho(v,q)).$$

If x is under the influence of v at level $n+1$ and $v \neq p, q$ then

$$c(\mu(v)\, K^{n+1})^{-\frac{1}{2}} \geqslant \min\,(\rho(v,p), \rho(v,q)).$$

By (2), as $p, q \in V_n$

$$\min\,(\rho(v,p), \rho(v,q)) \geqslant c'\mu(v)^{-\frac{1}{2}}\, K^{-\frac{1}{2}n}.$$

On comparing these last two inequalities we find a contradiction if $K > (c/c')^2$ which we shall assume henceforth.

If, on the contrary, $v = p$ (or q) then, by (2),

$$\begin{aligned} c(\mu(p)\, K^{n+1})^{-\frac{1}{2}} &\geqslant \min\,(\rho(x,p), \rho(x,q)) \\ &\geqslant \tfrac{1}{2}\epsilon\rho(p,q) \\ &\geqslant \tfrac{1}{2}\epsilon c'(\mu(p)\,\mu(q))^{-\frac{1}{2}}. \end{aligned}$$

Thus $\mu(q) \geqslant (2c/c'\epsilon)^2\, K^{n+1}$. This is impossible, as $q \in V_n$, if

$$K\left(\frac{\epsilon c'}{2c}\right)^2 > 1 \tag{4}$$

This we shall also assume.

Thus no point of $S(i_1, \ldots, i_n)$ is under the influence of $v \in W_n$ at level $n+1$. But we remarked above that, by (1), every point of S is under the influence of a point of W_{n+1} at level $n+1$. Thus every point of $S(i_1, \ldots, i_n)$ is under the influence of a point of V_{n+1} at level $n+1$. In other words, for each $x \in S(i_1, \ldots, i_n)$ there is $v \in V_{n+1}$ so that

$$\rho(v,x) < c(\mu(v)\, K^{n+1})^{-\frac{1}{2}}. \tag{5}$$

Further, for $v_1, v_2 \in W_n$, by (2) we have

$$\begin{aligned} \rho(v_1, v_2) &\geqslant c'(\mu(v_1)\,\mu(v_2))^{-\frac{1}{2}} \\ &\geqslant c'K^{-n}. \end{aligned} \tag{6}$$

Now, if $\epsilon < \frac{1}{2}$, when we use (2) and recall that $p, q \in V_n$, we have

$$|S(i_1, \ldots, i_n)| = (1-\epsilon)\,\rho(p,q) \tag{7}$$

$$\geqslant \tfrac{1}{2}c'K^{-n}. \tag{8}$$

We shall assume that $\epsilon < \frac{1}{2}$. Let

$$B(a,r) = \{x \in S \,|\, \rho(a,x) < r\}.$$

If $v \in V_{n+1}$ it influences at level $n+1$ the points of

$$B(v, c(\mu(v)\, K^{n+1})^{-\frac{1}{2}}) \subseteq B(v, cK^{-(n+\frac{1}{2})}). \tag{9}$$

Now we shall show that if

$$|S(i_1, \ldots, i_n)| \geqslant 7cK^{-(n+\frac{1}{2})}, \tag{10}$$

then $R(i_1, \ldots, i_n)$ has at least three points.

Recall that every point of $S(i_1, \ldots, i_n)$ is under the influence, at level $n+1$, of a point of V_{n+1}. Let ξ_0, ξ_1 be the end-points of $S(i_1, \ldots, i_n)$. If $v \notin S(i_1, \ldots, i_{n+1})$, $v \in V_{n+1}$ it can influence at level $n+1$, at most, by (9), in $S(i_1, \ldots, i_n)$, the set

$$S(i_1, \ldots, i_n) \cap B(\xi_0, cK^{-(n+\frac{1}{2})}) \cap B(\xi_1, cK^{-(n+\frac{1}{2})}).$$

Let $T(i_1, \ldots, i_n)$ be the complement in $S(i_1, \ldots, i_n)$ of this set. By (10)

$$|T(i_1, \ldots, i_n)| \geqslant 5cK^{-(n+\frac{1}{2})}. \tag{11}$$

Every point of $T(i_1, \ldots, i_n)$ is under the influence, at level $n+1$, of at point of

$$S(i_1, \ldots, i_n) \cap V_{n+1} = R(i_1, \ldots, i_n).$$

By (9) and (11) at least three points are required. Hence $R(i_1, \ldots, i_n)$ has at least three points if (10) is true. But this is so, from (8), if

$$K > (14c/c')^2,$$

which we shall assume. This shows that the inductive definition was valid.

Now we must establish some further properties. Let ξ_0, ξ_1 be the end-points of $S(i_1, \ldots, i_n)$ and let

$$R(i_1, \ldots, i_n) = \{v_1, v_2, \ldots, v_t\}.$$

We shall assume that there is a branch of arg on $\overline{S(i_1, \ldots, i_n)}$ so that

$$\arg\{\xi_0\} < \arg(v_1) < \arg(v_2) < \ldots < \arg(v_t) < \arg(\xi_1).$$

By (10) there are $\eta_0, \eta_1 \in S(i_1, \ldots, i_n)$ so that

$$\rho(\xi_0, \eta_0) = \rho(\xi_1, \eta_1) = (3c/2)\, K^{-(n+\frac{1}{2})}. \tag{12}$$

But η_0, η_1 are under the influence of $u_0, u_1 \in V_{n+1}$ at level $n+1$. Thus for $j = 0, 1$,

$$\begin{aligned}\rho(u_j, \eta_j) &\leqslant c(\mu(u_j)\, K^{n+1})^{-\frac{1}{2}} \\ &\leqslant cK^{-(n+\frac{1}{2})}.\end{aligned} \tag{13}$$

Thus, by (12) $\qquad u_j \in S(i_1, \ldots, i_n)$

so $\qquad u_j \in R(i_1, \ldots, i_n)$.

So by (12), (13)

$$\left.\begin{aligned}\rho(\xi_0, v_1) &\leqslant (5c/2)\, K^{-(n+\frac{1}{2})}, \\ \rho(\xi_1, v_t) &\leqslant (5c/2)\, K^{-(n+\frac{1}{2})}.\end{aligned}\right\} \tag{14}$$

Let a be an integer $0 < a < t$ and let

$$S(i_1, \ldots, i_n, a)$$

be the interval between v_a and v_{a+1} constructed as above. Then, by (7),

$$|S(i_1, \ldots, i_n, a)| = (1-\epsilon)\, \rho(v_a, v_{a+1}).$$

Thus, by (14),

$$\begin{aligned}|S(i_1, \ldots, i_n)| &= \rho(\xi_0, v_1) + \rho(v_1, v_2) + \ldots + \rho(v_{t-1}, v_t) + \rho(v_t, \xi_1) \\ &\leqslant 5cK^{-(n+\frac{1}{2})} + (1-\epsilon)^{-1} \sum_{a=1}^{t-1} |S(i_1, \ldots, i_n, a)|.\end{aligned}$$

By (8)

$$\sum_{a=1}^{t-1} \frac{|S(i_1, \ldots, i_n, a)|}{|S(i_1, \ldots, i_n)|} \geqslant 1 - \epsilon - \frac{10c}{c'} K^{-\frac{1}{2}}. \tag{15}$$

By (1), as the mid-point of $\overline{p_a\, p_{a+1}}$ is under the influence some $v \in V_{n+1}$ to level $n+1$, by (9)

$$\rho(p_a, p_{a+1}) \leqslant 2cK^{-(n+\frac{1}{2})}.$$

Thus

$$|S(i_1, \ldots, i_n, a)| \leqslant 2cK^{-(n+\frac{1}{2})}. \tag{16}$$

So, by (8),

$$\frac{|S(i_1, \ldots, i_n, a)|}{|S(i_1, \ldots, i_n)|} \leqslant \frac{4c}{c'} K^{-\frac{1}{2}}. \tag{17}$$

Likewise, by using the estimates in reverse,

$$\frac{|S(i_1, \ldots, i_n, a)|}{|S(i_1, \ldots, i_n)|} \geqslant \frac{c'}{4c} K^{-\frac{1}{2}}. \tag{18}$$

In particular, (18) shows that $t \leqslant (4c/c')\, K^{\frac{1}{2}}$ (19)

so each i_j runs through a uniformly bounded range. Also, if $a \neq b$, by (2) and (16),

$$\begin{aligned}\rho(S(i_1, \ldots, i_n, a),\, S(i_1, \ldots, i_n, b)) &\geqslant \tfrac{1}{2}\epsilon(\rho(p_a, p_{a+1}) + \rho(p_b, p_{b+1})) \\ &\geqslant \epsilon c' K^{-(n+1)} \\ &\geqslant \tfrac{1}{2}\epsilon c' c^{-1} K^{-\frac{3}{2}} |S(i_1, \ldots, i_n)|. \end{aligned} \tag{20}$$

Thus, in view of (17)–(20), theorem 1 of Beardon (1965) can be applied to show that if α is such that, for all $(i_1, \ldots, i_n)$,

$$\sum_{j=1}^{t-1} \left(\frac{|S(i_1, \ldots, i_n, j)|}{|S(i_1, \ldots, i_n)|}\right)^{\alpha} \geqslant 1,$$

then the Hausdorff dimension of

$$T = \bigcap_n \bigcup_{(i_1, \ldots, i_n)} S(i_1, \ldots, i_n)$$

is at least α. Actually, in Beardon (1965), this is stated only if t is constant but it is easy to check that it is also true if t is uniformly bounded; this is true by (19).

Suppose there are X, $\eta \in]0, 1[$, $F \in \boldsymbol{Z}_+$ and x_a for $1 \leqslant a \leqslant F$ so that

$$0 < x_a \leqslant X < 1, \quad \sum_{a=1}^{F} x_a > 1 - \eta.$$

Then, for s with $0 < s < 1$

$$\begin{aligned}\sum_{a=1}^{F} x_a^s - \sum_{a=1}^{F} x_a &= \sum_{a=1}^{F} \int_s^1 |\ln x_a|\, x_a^t \,\mathrm{d}t \\ &\geqslant |\ln X| \int_s^1 \sum_{a=1}^{F} x_a^t \,\mathrm{d}t \\ &\geqslant (1-s)\, |\ln X| \sum_{a=1}^{F} x_a.\end{aligned}$$

Thus if $s = 1 - |\ln X|^{-1} \eta (1-\eta)^{-1}$ then

$$\sum_{a=1}^{F} x_a^s \geqslant 1.$$

Applying this to our case, and using (15) and (17) we see that the dimension of T exceeds

$$1 - (\ln((c' K^{\frac{1}{2}})/(10c)))^{-1} \left(\frac{\epsilon + (10c/c')\, K^{-\frac{1}{2}}}{1 - \epsilon - (10c/c')\, K^{-\frac{1}{2}}}\right).$$

Apart from the requirements that K be sufficiently large and ϵ sufficiently small the only restriction on K and ϵ was (4), that is

$$K\left(\frac{\epsilon c'}{2c}\right)^2 > 1.$$

Choose

$$\epsilon = \frac{4c}{c'} K^{-\frac{1}{4}}.$$

Then the Hausdorff dimension of T exceeds

$$1 - O(1/(K^{\frac{1}{4}} \ln K)),$$

for K large.

On the other hand T is a subset of

$$\{x|\rho(x,p) \geqslant \epsilon c'(K^{n+1}\mu(p))^{-\frac{1}{2}}, p \in V_n \text{ (all } n)\} \subseteq \{x|\rho(x,p) \geqslant (\epsilon c'/K^{\frac{1}{2}})\,\mu(p)^{-1}, p \in V_n \text{ (all } n)\}$$
$$= \{x|\rho(x,p) \geqslant (\epsilon c'/K^{\frac{1}{2}})\,\mu(p)^{-1} \text{ all } p \in V\}.$$

This set has therefore dimension exceeding

$$1 - O(1/(K^{\frac{1}{2}} \ln K))$$

and this proves the theorem.

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